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# ON THE KERNELS OF THE LOWERING OPERATORS FOR RANKED POSETS

FUMIO HAZAMA

( Received December 2, 2000 )

Submitted by K. K. Azad

## Abstract

A problem on the structure of the kernels of the lowering operators for any ranked posets is proposed, and a complete solution to it is given in the case of the face posets of regular polytopes of any dimension. It is shown to be useful in the investigation of Hodge cycles on abelian varieties of CM-type.

## 1. Introduction

In [4], [5] we associated abelian varieties of CM-type to various hyperplane arrangements, and investigated the structure of the rings of Hodge cycles on them. There the notion of *N-dominatedness* of an abelian variety is introduced, and plays a certain role for us to recognize what kind of nondivisorial Hodge cycles appears as a basic obstruction for abelian varieties to be nondegenerate in the sense of [8]. On the other hand, the investigation in [4] suggests that the notion could be interpreted in terms of combinatorics of certain posets attached to hyperplane arrangements. In the present paper, we focus on the combinatorial aspect, and show that the problem can be formulated and investigated most naturally in terms of arbitrary *ranked posets*.

2000 Mathematics Subject Classification: 14C30, 06A11.

Key words and phrases: ranked poset, lowering operator, abelian variety, Hodge cycle.

Specifically, we are interested in the structure of the kernels of the *lowering operators* acting on $\mathbf{Q}$-vector spaces attached to any ranked posets, and give a detailed analysis for the case of the *face posets* of regular polytopes of any dimension. As an application, we will show that we can attach a *degenerate* abelian variety to each of the face posets too, and determine the minimal $N$ such that it is $N$-dominated, using our combinatorial result.

The plan of the paper is as follows. In Section 2, we recall the notion of the lowering operator attached to a ranked poset, investigated in the context of *differential posets* in [12] and of $sl_2$*-posets* in [9], and formulate our problem in terms of the kernel of the lowering operator. In Section 3, we investigate the structure of the kernel for each of the face posets of regular polytopes. Interestingly, the representation theory of the Lie algebra $sl_2$ plays a definite role in our study of the three infinite families of regular polytopes. Section 4 describes our method of construction of abelian varieties of CM-type associated to these face posets and investigate the $N$-dominatedness of them.

## 2. Lowering Operators: Problem Setting

In this section, we give some definitions and notation, which will be used throughout the paper, and formulate the main problem.

Let $P$ be a ranked poset with rank function $\rho : P \to \{0, 1, \ldots, n\}$ such that $\rho(x) = 0$ if $x$ is a minimal element of $P$, and $\rho(y) = \rho(x) + 1$ if $y$ covers $x$ in $P$. Sometimes when convenient we extend the poset by adding to it a unique minimum element which is regarded as having rank $-1$. All the posets considered in this paper are assumed to be *finite*. We write $P_i$ for $\rho^{-1}(i) \subset P$, the subset of elements of rank $i$, and let $p_i = \# P_i$ denote the number of elements of $P_i$. We fix once for all a numbering $x(i)_1, \ldots, x(i)_{p_i}$ of the elements of $P_i$. Let $V(P_i)$, $i = 0, \ldots, n$, denote the $\mathbf{Q}$-vector space consisting of formal linear combinations $\sum_{1 \le j \le p_i} a_j x(i)_j$, $a_j \in \mathbf{Q}$, and let $V(P_{-1}) = \{0\}$ when we take $P_{-1}$ into account.

Furthermore, we put $V(P) = \bigoplus_i V(P_i)$. Let $D_i : V(P_i) \to V(P_{i-1})$, $i = 1, \ldots, n$, denote the linear map defined by the property $D_i(x) = \sum_{y \text{ is covered by } x} y \in V_{i-1}$, and let $D_0$ be the zero map. We call $D_i$ the *lowering operator at level i*. Putting $D_i$, $i = 0, \ldots, n$ together, we let $D = \bigoplus_i D_i : V(P) \to V(P)$ and call it the lowering operator. Dually we introduce an operator $U_i : V(P_i) \to V(P_{i+1})$, $i = 0, \ldots, n-1$, by $U_i(x) = \sum_{x \text{ is covered by } y} y \in V_{i+1}$, and call it the *raising operator at level i*. If we represent $D_i$ with respect to the natural basis $x(i)_1, \ldots, x(i)_{p_i} \in V_i$ and $x(i-1)_1, \ldots, x(i-1)_{p_{i-1}} \in V_{i-1}$, then we obtain a $p_{i-1} \times p_i$ matrix $M_i$, which is a 0-1-matrix. We call this matrix *the incidence matrix at level i*. We say that an element $\nu = \sum_{1 \le j \le p_i} a_j x(i)_j \in V_i$ is a $\pm 1$*-vector*, when $a_j \in \{0, \pm 1\}$ for all $j$, and define the *height* $h(\nu)$ of $\pm 1$-vector $\nu$ to be $\left( \sum_{1 \le j \le p_i} |a_j| \right) \Big/ 2$.

The problem which mainly concerns us in this paper is the following:

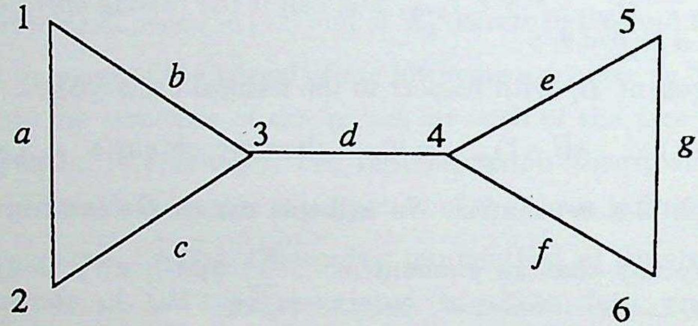
(1) For what kind of ranked posets is the kernel of the lowering operator $D_i$ spanned by $\pm 1$-vectors?

For a poset $P$ for which the kernel of the lowering operator $D_i$ are spanned by $\pm 1$-vectors, we introduce the number $d(P_i) = \min_B \max_\nu \{h(\nu);\ \nu \in B\}$, where $B$ runs through the set of bases of the kernel of $D_i$ consisting wholly of $\pm 1$-vectors. We call $d(P_i)$ the *dominating height at level i of the poset P*. Another problem which we are interested in is the following:

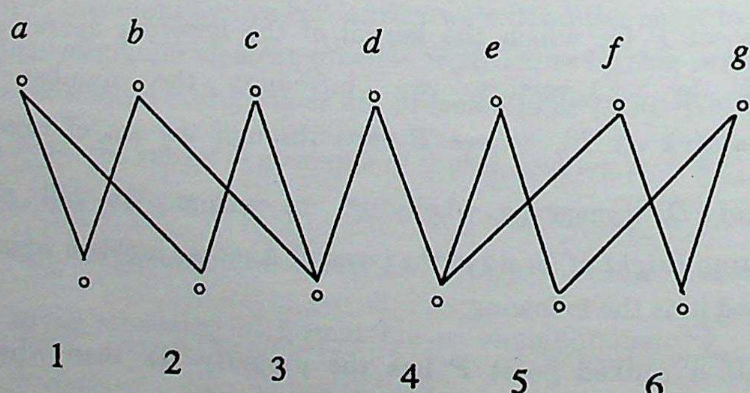
(2) If a ranked poset $P$ has the property (1), then what is the dominating height at level $i$ of $P$?

**Remark.** Roughly speaking, if we can associate an abelian variety $A$ of CM-type to a poset in such a way as in Section 4, then the dominating height at level $i$ of the poset gives the upper bound of codimension of Hodge cycles on the abelian variety which are needed to generate the whole ring of Hodge cycles of any power $A^{\ell}$, $\ell \geq 1$.

**Example 2.1.** Let $G = (V, E)$ be the graph




with the set of vertices $V = \{1, 2, 3, 4, 5, 6\}$, and the set of edges $E = \{a, b, c, d, e, f, g\}$. Its incidence structure allows one to obtain a poset $P = P_0 \coprod P_1$ of rank one, where $P_0 = V$, $P_1 = E$, with Hasse diagram:



The incidence matrix $M_1$ at level one is given by

$$M_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

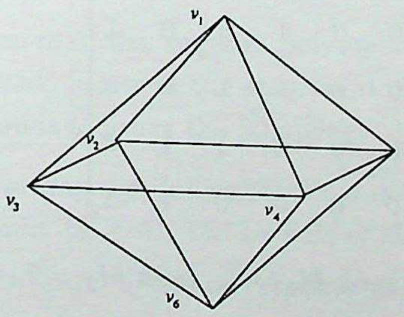
Its kernel is easily found to be one-dimensional and generated by the column vector ${}^t(1\ \ -1\ \ -1\ \ 2\ \ -1\ \ -1\ \ 1)$. Therefore, this poset does not have the property (1).

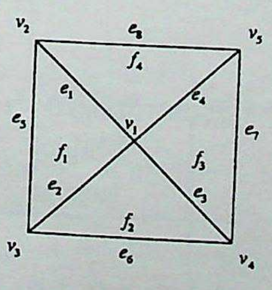
**Example 2.2.** Let $P = P_0 \coprod P_1 \coprod P_2$ be the poset which arises naturally from the regular tetrahedron with

$P_0 = \{v_i;\ 1 \le i \le 6\}$, (*vertices*)

$P_1 = \{e_j;\ 1 \le j \le 12\}$, (*edges*)

$P_2 = \{f_k;\ 1 \le k \le 8\}$. (*faces*)

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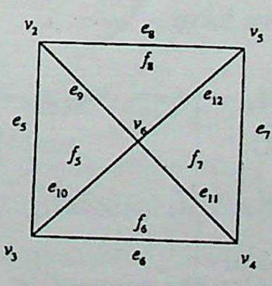



lower half

The incidence matrices at level 1 and 2 are given by

$$M_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

One checks easily that $\operatorname{rank} M_1 = 6$, $\operatorname{rank} M_2 = 7$, and that the kernel of $M_1$ (resp. $M_2$) is generated by $\pm 1$-vectors of weight two (resp. four). Specifically, we have

$$\ker M_1 = \left\langle e_{1\bar{3}\bar{5}6}, e_{2\bar{4}\bar{6}7}, e_{2\bar{4}\bar{5}8}, e_{1\bar{2}\bar{9}(10)}, e_{1\bar{3}\bar{9}(11)}, e_{1\bar{4}\bar{9}(12)} \right\rangle_{\mathbf{Q}},$$

$$\ker M_2 = \left\langle f_{1\bar{2}3\bar{4}\bar{5}6\bar{7}8} \right\rangle_{\mathbf{Q}},$$

where we employ the convention $e_{i\bar{j}\bar{k}l} = e_i - e_j - e_k + e_l$, and so on.

These two examples suggest us to form the following working hypothesis:

In order that the kernels of incidence matrices of a poset at any levels are spanned by $\pm 1$-vectors, the poset should posses a certain symmetry. Thus we are led to the investigation of the posets arising from *regular polytopes.* In the next section, we will see that for any regular polytopes (with two possible exceptions), the corresponding posets do have the property (1) above, and will determine the minimum of the weights of the $\pm 1$-vectors spanning the kernels.

**Remark.** After the author wrote up the first draft of the paper, he noted that a related problem was considered already in [7] in the context of the representation theory of the general linear groups over finite fields. As far as the author knows, however, there are no other articles than ours that treat our problems from a unified viewpoint.

## 3. Regular Polytopes and their Lowering Operators

In this section, we investigate the kernel of the lowering operator for the poset associated to each of the regular polytopes.

It is well known that the regular polytopes of any dimension are classified as in the table below in the statement of Theorem 3.1 ([1]). The purpose of this section is to prove the following:

**Theorem 3.1.** *For any regular polytopes of dimension n with possible exceptions* $\{3, 3, 5\}$ *and* $\{5, 3, 3\}$, *the kernels of the lowering operators at any level i with* $1 \leq i \leq n/2$ *of the corresponding posets are generated by* $\pm 1$*-vectors. Moreover, their dominating heights are given in the following table*:

| Dimension | Schlafli symbol | Name | Dominating height |
|---|---|---|---|
| 2 | $\{m\}$ | Regular $n$-gon | $n/2$, if $n$ is even<br>0, if $n$ is odd |

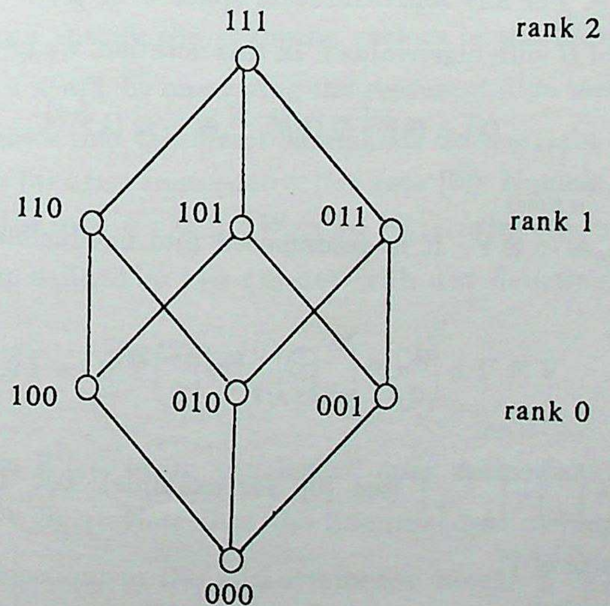
| | | | |
|---|---|---|---|
| 3 | $\{3, 3\}$ | Tetrahedron | 2 (level 1) |
| | $\{3, 4\}$ | Octahedron | 2 (level 1) |
| | $\{4, 3\}$ | Cube | 2 (level 1) |
| | $\{5, 3\}$ | Dodecahedron | 4 (level 1) |
| | $\{3, 5\}$ | Icosahedron | 2 (level 1) |
| 4 | $\{3, 3, 3\}$ | 5-cell $\alpha_4$ | 2 (level 1) |
| | | | 0 (level 2) |
| | $\{3, 3, 4\}$ | 16-cell $\beta_4$ | 2 (level 1) |
| | | | 0 (level 2) |
| | $\{4, 3, 3\}$ | Tessaract $\gamma_4$ | 2 (level 1) |
| | | | 0 (level 2) |
| | $\{3, 4, 3\}$ | 24-cell | 2 (level 1) |
| | | | 4 (level 2) |
| | $\{3, 3, 5\}$ | 600-cell | ? |
| | $\{5, 3, 3\}$ | 120-cell | ? |
| $n \geq 5$ | $\{3^{n-1}\}$ | Regular simplex $\alpha_n$ | $2^i$ (level $i$) |
| | $\{3^{n-2}, 4\}$ | Cross polytope $\beta_n$ | $2^{2i-1}$ (level $i$) |
| | $\{4, 3^{n-2}\}$ | Measure polytope $\gamma_n$ | $2^{2i-1}$ (level $i$) |

We will treat each of them separately in a series of subsections. Before we start our analysis of each polytope, we fix some notation. We denote by $N_0$ the number of vertices, $N_1$ the number of edges, and so on, therefore by $N_k$ the number of $k$-dimensional faces of a polytope in

question. We begin with the three infinite families, namely regular simplexes, cross polytopes, and measure polytopes.

### 3.1. Regular Simplex $\alpha_n$

The regular $n$-simplex $\alpha_n$ in $n$-space $\mathbf{R}^n$ is defined as a finite region of $\mathbf{R}^n$ enclosed by $n+1$ hyperplanes so that the $n(n+1)/2$ edges are all equal. The corresponding poset $P(\alpha_n)$ is isomorphic to the *Boolean lattice* $B_{n+1}$ of all subsets of $\{1, \ldots, n+1\}$ ordered naturally by inclusion relation. Notice that the automorphism group of $P(\alpha_n)$ is isomorphic to the symmetric group $S_{n+1}$ of degree $n+1$, and it acts on the regular $n$-simplex *flag-transitively*. In order to express the elements of $P(\alpha_n)$, we identify a subset $S$ of $\{1, \ldots, n+1\}$ with its characteristic function $\chi_S : \{1, \ldots, n+1\} \to \{0, 1\}$, and represent it as a row $s_1 s_2 \cdots s_{n+1}$ of 0 and 1 defined by $s_i = \chi_S(i)$, $i = 1, \ldots, n+1$. The following figure illustrates the poset $P(\alpha_2)$:



Note that the rank of a face corresponding to a subset $S \subset \{1, \ldots, n+1\}$ is equal to $\# S - 1$. The lowering operator $D_1 : V(P(\alpha_2)_1) \to V(P(\alpha_2)_0)$ at level one is given by

$$D_2 : 110 \mapsto 100 + 010, \quad 101 \mapsto 100 + 001, \quad 011 \mapsto 010 + 001.$$

This suggests us that the lowering operators at any level, in general are intimately related with the representation of the Lie algebra $sl_2$. Specifically, let $\mathfrak{g} = sl_2(\mathbf{Q})$ and let $V = \mathbf{Q}^2$ be the natural representation space of $\mathfrak{g}$. As standard generators of $\mathfrak{g}$, we take

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let $(k)$ denote the irreducible representation of $\mathfrak{g}$ afforded by the space of symmetric tensors $S^k(V)$ of degree $k$, $k \geq 0$. (We employ the convention that (0) means that trivial representation.) It is the direct sum of one-dimensional eigenspaces of $H$ with weights $k, k-2, k-4, \ldots, -(k-2), -k$. For any representation space $U$ of $\mathfrak{g}$, let $U^{[i]}$ denote the eigenspace of $H$ with eigenvalue $i$. In this notation, we have

$$(k) = (k)^{[k]} \oplus (k)^{[k-2]} \oplus \cdots \oplus (k)^{[-k]}. \tag{3.1}$$

Let $W_n = \overbrace{V \otimes \cdots \otimes V}^{n \text{ times}}$. It is decomposed into irreducible components as follows:

$$W_n \cong \bigoplus_{0 \leq i \leq n/2} m_i(n-2i), \tag{3.2}$$

where $m_i = \binom{n}{i} - \binom{n}{i-1}$ (see [6], for example). Let $\overline{1}$, $\overline{0}$ denote the standard basis of $V$:

$$\overline{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \overline{0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then, as a basis of the $2^n$-dimensional space $W_n$, we can take $\overline{\varepsilon}_1 \otimes \cdots \otimes \overline{\varepsilon}_n$, $\varepsilon_i \in \{0, 1\}$, which we will denote by the juxtaposition $\overline{\varepsilon}_1 \cdots \overline{\varepsilon}_n$. The following lemma (whose proof is easy) justifies our notation:

**Proposition 3.1.1.** *There is an isomorphism* $\varphi_i : V(P(\alpha_n)_i) \to (W_{n+1})^{[2i-n+1]}$, $0 \le i \le n$, *which sends* $\varepsilon_1 \cdots \varepsilon_{n+1}$ *to* $\overline{\varepsilon}_1 \cdots \overline{\varepsilon}_{n+1}$ *such that it intertwines the lowering operator* $D_i$ *and* $Y\big|_{(W_{n+1})^{[2i-n+1]}}$, *namely,* $\varphi_{i-1} \circ D_i = Y\big|_{(W_{n+1})^{[2i-n+1]}} \circ \varphi_i$. *Therefore, if we put* $\varphi = \bigoplus_i \varphi_i : V \to W_n$, *then we have* $\varphi \circ D = Y \circ \varphi$.

Thus our problem is reduced to the investigation of the kernel of $Y \in \mathscr{g}$ regarded as an operator on $W_{n+1}$. In other words, we are to find the lowest weight vectors in each of irreducible constituents of $W_{n+1}$. Thus it directly follows from (3.1) and (3.2) that

(3.3) the lowering operator $D_i$ is surjective for $1 \le i \le n/2$, and injective for $n/2 < i \le n$.

Moreover, we can specify the spanning vectors of the kernel of $D_i$ for each $i$ with $1 \le i \le n/2$ by employing the representation theory of $sl_2$ as follows. One knows that the direct summands on the right hand side of (3.2) other than $(n)$ arise from *contraction* (see [6]). Namely, they belong to the image of the map $\sigma \circ \lambda_{n-1} : W_{n-1} \to W_{n+1}$, where $\lambda_j : W_j \to W_{j+2}$ is the linear map defined by $t \mapsto t \otimes \mathbf{det}$ with $\mathbf{det}$ denoting the 2-tensor

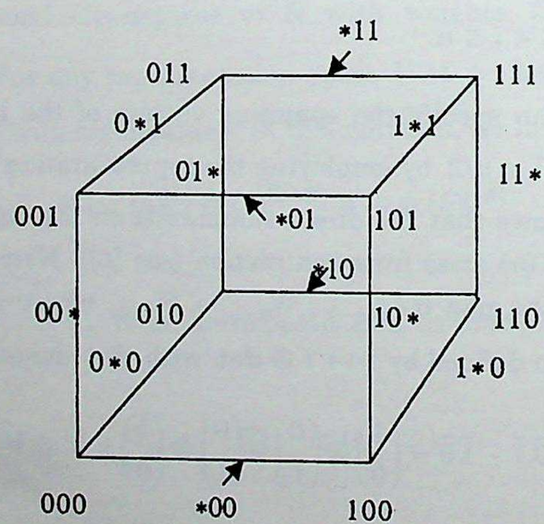
$$\overline{0}\,\overline{1} - \overline{1}\,\overline{0} = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \in V \otimes V,$$

and $\sigma : W_{n+1} \to W_{n+1}$ being the linear map defined naturally by a permutation $\sigma \in S_{n+1}$. Note that the 2-tensor $\mathbf{det}$ corresponds, under the natural projection, to the anti-symmetric tensor $2 \cdot \overline{0} \wedge \overline{1} \in V \wedge V$, which is $sl_2$-invariant. Noticing that the map $\lambda_j : W_j \to W_{j+2}$ leaves the

height of a tensor invariant, we see that the kernel of $Y\Big|_{(W_{n+1})^{[2i-n+1]}}$ coincides with the sum of the image of the composed map $\sigma \circ \lambda_{n-1} \circ \lambda_{n-3} \circ \cdots \circ \lambda_{n+1-2i} : (W_{n+1-2i})^{[2i-n+1]} \to (W_{n+1})^{[2i-n+1]}$ with $\sigma \in S_n$ running through all the permutations. Thus we complete the proof of Theorem 3.1 in the case of regular simplexes, since the height of $\mathbf{det}^{\otimes i}$ is equal to $2^{i-1}$.

### 3.2. Measure Polytope $\gamma_n$

This is the hypercube of dimension $n$. We may assume its vertices consist of $(\varepsilon_1, \ldots, \varepsilon_n) \in \mathbf{R}^n$, $\varepsilon_i \in \{0, 1\}$. Two vertices constitute an edge if and only if their *Hamming distance* is equal to one, namely their coordinates differ in exactly one position. In order to express a line, we introduce the notation $*$ to represent the place where the difference occurs. The following figure of the usual cube clarifies the situation:




In order to deal with higher dimensional cases, we introduce the notation $\varepsilon_1 \cdots \varepsilon_n$ with several $\varepsilon_i$ equal to $*$: If $\varepsilon_i = *$ with $i$ contained in a subset $S \subset \{1, \ldots, n\}$ and $\varepsilon_i \in \{0, 1\}$ for $i \in \{1, \ldots, n\} - S$, then $\varepsilon_1 \cdots \varepsilon_n$

represents the $\#(S)$-dimensional face of the hypercube defined by $x_i = \varepsilon_i$, $i \in \{1, ..., n\} - S$. For such $\varepsilon_1 \cdots \varepsilon_n$, let $S(\varepsilon_1 \cdots \varepsilon_n)$ denote the set $\{i \in \{1, ..., n\};\ \varepsilon_i = *\}$. In this terminology, the poset $P(\gamma_n)$ is expressed as

$$P(\gamma_n) = \coprod_{-1 \le i \le n} P_i,$$

$$P_i = \{\varepsilon_1 \cdots \varepsilon_n \in \{0, 1, *\}^n;\ \#(S(\varepsilon_1 \cdots \varepsilon_n)) = i\},\ 0 \le i \le n,$$

$$P_{-1} = \{\phi\} \text{ with } \phi \text{ the unique minimal element.}$$

We see from this description that the number $N_i$ of the $i$-dimensional faces is equal to $N_i = 2^{n-i}\binom{n}{i}$ (see [1] for more complete treatment). The lowering operator $D_i$ for the poset $P(\gamma_n)$ at level $i$ is given simply by

$$D_i(\varepsilon_1 \cdots \varepsilon_n) = \sum_{i \in S(\varepsilon_1 \cdots \varepsilon_n)} \left\{ \left( \varepsilon_1 \cdots \overset{i}{\breve{0}} \cdots \varepsilon_n \right) + \left( \varepsilon_1 \cdots \overset{i}{\breve{1}} \cdots \varepsilon_n \right) \right\}.$$

Inspired by this equality, we endow the three-dimensional $\mathbf{Q}$-vector space $V = \mathbf{Q}^3$ with an action of the Lie algebra $sl_2(\mathbf{Q})$ as follows. Let

$$X_3 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\quad H_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & -1/2 \\ 0 & -1/2 & -1/2 \end{pmatrix}.$$

Then the linear map $\rho : sl_2 \to EndV$ defined by $\rho(X) = X_3$, $\rho(Y) = Y_3$, $\rho(H) = H_3$ is a homomorphism of Lie algebras, since

$$[X_3, Y_3] = H_3,\quad [H_3, X_3] = 2X_3,\quad [H_3, Y_3] = -2Y_3.$$

Let $\overline{*}$, $\overline{0}$, $\overline{1}$ denote the standard basis of $V$:

$$\overline{*} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\quad \overline{0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\quad \overline{1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then the generators $\{X_3, Y_3, H_3\}$ of $\rho(sl_2) \subset End V$ act on them as follows:

$$X_3 : \overline{*} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \overline{0} \mapsto (1/2)\overline{*}, \qquad \overline{1} \mapsto (1/2)\overline{*},$$

$$Y_3 : \overline{*} \mapsto \overline{0} + \overline{1}, \qquad \overline{0} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \overline{1} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$H_3 : \overline{*} \mapsto \overline{*}, \qquad \overline{0} \mapsto (-1/2)(\overline{0} + \overline{1}), \qquad \overline{1} \mapsto (-1/2)(\overline{0} + \overline{1}).$$

Hence the eigenvalues of $H_3$ are 1, $-1$, and 0, and the corresponding generators of the eigenspaces are $\overline{*}$, $\overline{0} + \overline{1}$, and $\overline{0} - \overline{1}$, respectively. Therefore, the $sl_2$-module $V$ is decomposed into irreducible components as

$$V = (1) \oplus (0), \tag{3.4}$$

the direct sum of the defining representation and the trivial one.

Therefore, if we set $T_n = \overbrace{V \otimes \cdots \otimes V}^{n\ times}$ and represent a tensor by the juxtaposition, then we have the following:

**Proposition 3.2.1.** *There is an isomorphism* $\varphi : V(\gamma_n) \to T_n$ *which sends* $\varepsilon_1 \cdots \varepsilon_n$ *to* $\overline{\varepsilon}_1 \cdots \overline{\varepsilon}_n$ *such that it intertwines the lowering operator* $D$ *and* $Y_3$, *namely*, $\varphi \circ D = Y_3 \circ \varphi$.

It follows from (3.4) that $T_n$ is decomposed as an $sl_2$-module into irreducible components as

$$T_n \cong ((1) \oplus (0))^{\otimes n} \cong \bigoplus_{0 \le k \le n} \binom{n}{k} ((1)^{\otimes k} \otimes (0)^{n-k}) \cong \bigoplus_{0 \le k \le n} \binom{n}{k} A_k, \tag{3.5}$$

where $A_k \cong \bigoplus_{0 \le i \le k/2} m_i(k - 2i) (\cong (1)^{\otimes k})$, which appears in the previous subsection. Thus our task is to find the lowest weight vectors in various

irreducible components appearing in (3.5). Note that the vector $\overline{0} - \overline{1}$ spans the summand (0) in (3.4), and that the vector $\overline{0} + \overline{1}$ spans the (one-dimensional) subspace of the lowest vectors in (1). Let $V_-$ denote the submodule consisting of the lowest vectors for any $sl_2$-module $V$. It follows from (3.5) that the subspace $(T_n)_-$ of the lowest vectors in $T_n$ is expressed as

$$(T_n)_- \cong \bigoplus_{0 \le k \le n} \binom{n}{k} (((1)^{\otimes k})_- \otimes (0)^{n-k}). \tag{3.6}$$

This time the anti-symmetric $sl_2$-invariant tensor of degree two is given by

$$\mathbf{det} = \overline{*} \otimes (\overline{0} + \overline{1}) - (\overline{0} + \overline{1}) \otimes \overline{*} \in V \otimes V.$$

By a similar argument to the one given in the previous subsection, we have

$$\ker\left(Y \Big|_{\varphi(V(\gamma_n)_i)}\right) = \sum_{\sigma \in S_n} \sigma(\mathrm{Im}(\lambda_{n-2} \circ \lambda_{n-4} \circ \cdots \circ \lambda_{n-2i} : T'_{n-2i} \to T_n)),$$

where $T'_k = V'^{\otimes k}$, $V' = \langle \overline{0}, \overline{1} \rangle_{\mathbf{Q}} \subset V$. Therefore, expanding

$$\mathbf{det}^{\otimes i} = (\overline{*} \otimes (\overline{0} + \overline{1}) - (\overline{0} + \overline{1}) \otimes \overline{*})^{\otimes i},$$

we obtain the assertion of the theorem in this case.

### 3.3. Cross Polytope $\beta_n$

This is the *dual* of the measure polytope in the sense that the corresponding poset $P(\beta_n)$ is obtained from reversing the partial order defining the poset $P(\gamma_n)$. Therefore, by denoting a dual of each original face of $\gamma_n$ by the same symbol as in the case of the measure polytope, we have the decomposition

$$P(\beta_n) = \coprod_{-1 \le i \le n} P_i,$$

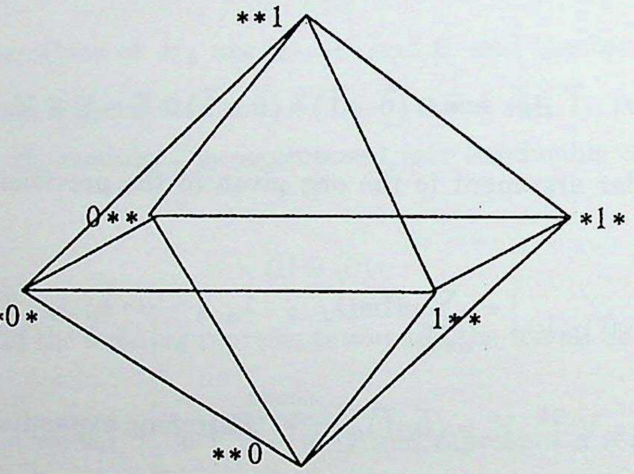
where

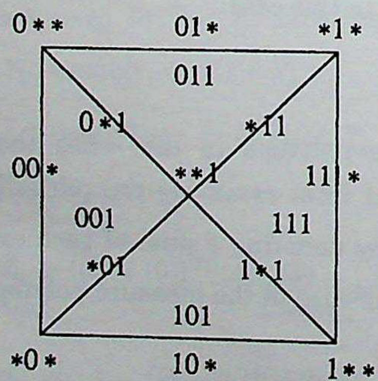
$$P_i = \{\varepsilon_1 \cdots \varepsilon_n \in \{*,\ \mathbf{0},\ \mathbf{1}\};\ \# S(\varepsilon_1 \cdots \varepsilon_n) = n - 1 - i\}, \quad -1 \le i \le n-1,$$

$$P_n = \{\Omega\} \text{ with } \Omega \text{ the unique maximal element.}$$

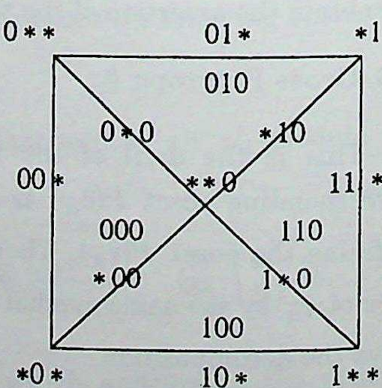
Hence the number $N_i$ of the $i$-dimensional faces is equal to $2^{i+1}\binom{n}{i+1}$.

We illustrate the situation by the case $n = 3$, namely by the case of the regular octahedron:




*the upper half*




*the lower half*

Noting that the vector space $V(\beta_n)_i$ is isomorphic to the dual vector space $V(\gamma_n)_{n-1-i}^*$ of $V(\gamma_n)_{n-1-i}$, we see that the lowering operator $D$ for $V(\beta_n)$ is identified with the dual map of the one for $V(\gamma_n)$, and that it is given by

$$D(\varepsilon_1 \cdots \varepsilon_n) = \sum_{i \in \{1, \ldots, n\} - S(\varepsilon_1 \cdots \varepsilon_n)} \left( \varepsilon_1 \cdots \overset{i}{\overset{\smile}{*}} \cdots \varepsilon_n \right), \tag{3.7}$$

which is also expected from the figure given above. Inspired by the result in the previous subsection, we endow the $\mathbf{Q}$-vector space $V' = \mathbf{Q}^3$ with an action of the Lie algebra $sl_2(\mathbf{Q})$ as follows. Let

$$X_3' = \begin{pmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}, \quad Y_3' = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_3' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

Then one can check that the linear map $\rho' : sl_2 \to EndV'$ defined by $\rho'(X) = X_3'$, $\rho'(Y) = Y_3'$, $\rho'(H) = H_3'$ is a homomorphism of Lie algebras. Let

$$\overline{*} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \overline{0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \overline{1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then the generators $X_3'$, $Y_3'$, $H_3'$ of $\rho'(sl_2) \subset EndV'$ act on them as follows:

$$X_3' : \overline{*} \mapsto (1/2)(\overline{0} + \overline{1}), \qquad \overline{0} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \overline{1} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$Y_3' : \overline{*} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \overline{0} \mapsto \overline{*}, \qquad \overline{1} \mapsto \overline{*},$$

$$H_3 : \overline{*} \mapsto (-1)\overline{*}, \qquad \overline{0} \mapsto (1/2)(\overline{0} + \overline{1}), \qquad \overline{1} \mapsto (1/2)(\overline{0} + \overline{1}).$$

Note that the eigenvalues of $H_3$ are $-1$, $1$, and $0$, and the corresponding generators of the eigenspaces are $\overline{*}$, $\overline{0}+\overline{1}$, and $\overline{0}-\overline{1}$, respectively. Hence the $sl_2$-module $V'$ is decomposed into irreducible components as

$$V' = (1) \oplus (0), \tag{3.8}$$

the sum of the defining representation and the trivial one. Let $T'_n = \overbrace{V' \otimes \cdots \otimes V'}^{n\,times}$. Then we have the following:

**Proposition 3.3.1.** *There is an isomorphism* $\varphi' : V(\beta_n) \to T'_n$ *which sends* $\varepsilon_1 \cdots \varepsilon_n$ *to* $\overline{\varepsilon}_1 \cdots \overline{\varepsilon}_n$ *such that it intertwines the lowering operator* $D$ *and* $Y'_3$, *namely,* $\varphi' \circ D = Y'_3 \circ \varphi'$.

Once we have obtained this translation of our problem into the one in the representation theory of $sl_2$, our proof proceeds as in the previous subsection. Thus we obtain the desired result. (Details are left to the reader.)

### 3.4. Regular Polygons

The matrix $M_1$ for the lowering operator at level one for the regular $n$-gon is a square matrix of degree $n$, and is given by

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}$$

for an appropriate numbering of vertices and edges. Therefore, the rank of $M_1$ is $n-1$ (resp. $n$) when $n$ is even (resp. odd). When $n$ is even, its kernel is one-dimensional and generated by the column vector ${}^t(1\ -1\ 1\ \cdots\ 1\ -1)$. Thus it is generated by $\pm 1$-vector of height $n/2$, and hence the proof is complete in the case of regular polygon.

### 3.5. Tetrahedron, Cube, Octahedron

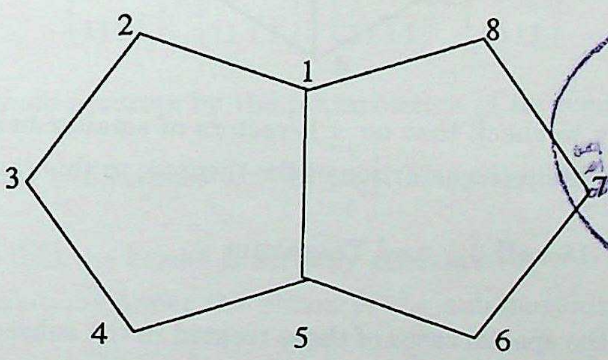
These are special cases of $\alpha_n$, $\gamma_n$, and $\beta_n$ treated in the subsections 3.1, 3.2, and 3.3, respectively. Therefore, nothing remains.

### 3.6. Dodecahedron $\mathcal{D}$

It is quite easy to write down the representation matrices $M_1$, $M_2$ of the lowering operators at level one and two, respectively, and to compute their ranks and kernels. We record here the results only. For $M_1$, which is a $20 \times 30$ matrix, we obtain

$$\operatorname{rank} M_1 = 20, \quad \dim \ker M_1 = 10,$$

and the kernel is generated by the $\pm 1$-vector of height four $12 - 23 + 34 - 45 + 56 - 67 + 78 - 81$ (see the figure below, where we draw a pair of neighboring faces) together with its translations under the action of the dodecahedral group:






Since it is easy to check that no $\pm 1$-vectors of smaller height belong to the kernel, we obtain the assertion of the theorem in this case. For $M_2$, which is a $30 \times 12$-matrix, one can easily check that it is of full-rank. Hence $\ker M_2 = \{0\}$. Thus our proof for the dodecahedron is complete.

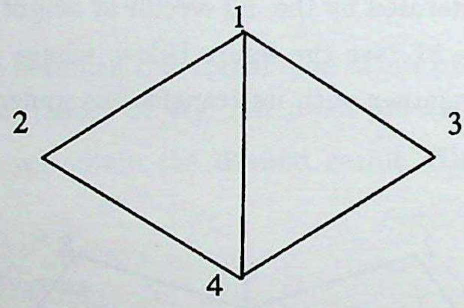
### 3.7. Icosahedron $\mathcal{I}$

This is the dual of the dodecahedron. Therefore, if we denote the

corresponding poset by $P$, and the associated vector space by $V(P)$, then we have the decomposition $V(P) = V_0 \oplus V_1 \oplus V_2$, with

$$\dim V_0 = 12, \quad \dim V_1 = 30, \quad \dim V_2 = 20.$$

Noting that the lowering operators $D_1$, $D_2$ are the dual of those for the dodecahedron, we have $\operatorname{rank} D_1 = 12$, $\operatorname{rank} D_2 = 20$. Moreover, one can check by a direct computation that the kernel of $D_1$ is generated by the $\pm 1$-vector of height two $12 - 23 + 34 - 41$ (see figure, where we draw a pair of neighboring faces) together with its translations under the action of the icosahedral group:




Since it is easy to check that no $\pm 1$-vectors of smaller height belong to the kernel, we obtain the assertion of the theorem in this case.

### 3.8. 5-cell $\alpha_4$, 16-cell $\beta_4$, and Tessaract $\gamma_4$

These are the special cases of those treated in the subsections 3.1, 3.3, and 3.2, respectively. Hence we are already done.

### 3.9. 24-cell {3, 4, 3}

This regular polytope is realized as the convex hull of 24-points

$$(\pm 1, \pm 1, \pm 1, \pm 1)$$

and

$$(\pm 2, 0, 0, 0), (0, \pm 2, 0, 0), (0, 0, \pm 2, 0), (0, 0, 0, \pm 2)$$

in the four-dimensional Euclidean space $\mathbf{R}^4$ (see [1]). The supporting hyperplanes of its 3-dimensional faces are given by $\pm x_i, \pm x_j = 2$, $1 \le i < j \le 4$. We denote by $H_{ab}$, $a, b \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$, with $|a| \neq |b|$, the hyperplane $\operatorname{sgn}(a) x_{|a|} + \operatorname{sgn}(b) x_{|b|} = 2$. Each of the 2-dimensional faces is defined as the intersection of a pair of hyperplanes $H_{ab}, H_{cd}$ with $\#(\{a, b\} \cap \{c, d\}) = 1$, which we denote by $(ab, cd)$. It follows that the numbers $N_i$ of $i$-dimensional faces are given by

$$N_0 = 24, \quad N_1 = 96, \quad N_2 = 96, \quad N_3 = 24.$$

### 3.9.1. The Lowering Operator at Level One:

This is represented by a $24 \times 96$-matrix $M_1$. One can check by a tedious computation that $\operatorname{rank} M_1 = 24$ and that a typical element of its kernel is given by the $\pm 1$-vector of height two

$$\begin{pmatrix} 1111 \\ 111\overline{1} \end{pmatrix} - \begin{pmatrix} 111\overline{1} \\ 11\overline{1}\,\overline{1} \end{pmatrix} + \begin{pmatrix} 11\overline{1}\,\overline{1} \\ 11\overline{1}1 \end{pmatrix} - \begin{pmatrix} 11\overline{1}1 \\ 1111 \end{pmatrix},$$

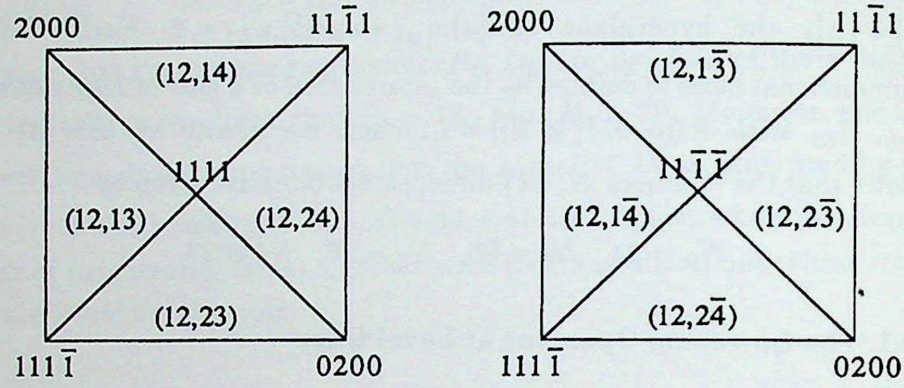
where we denote a vertex by the juxtaposition of its coordinates and an edge connecting two vertices $\varepsilon, \varepsilon'$ by $\begin{pmatrix} \varepsilon \\ \varepsilon' \end{pmatrix}$. Moreover, one can check (after some efforts) that the kernel is actually generated by this vector together with its translations under the action of the automorphism group of the 24-cell.

### 3.9.2. The Lowering Operator at Level Two:

This is represented by a $96 \times 96$-matrix $M_2$. One can check by a tedious computation that $\operatorname{rank} M_2 = 73$, and hence $\dim \ker M_2 = 23$. A typical element of its kernel is found to be the $\pm 1$-vector of height four,

$$\begin{aligned} &(12, 13) - (12, 23) + (12, 24) - (12, 14) \\ &\quad + (12, 2\overline{4}) - (12, 2\overline{3}) + (12, 1\overline{3}) - (12, 1\overline{4}). \end{aligned}$$

The situation is illustrated in figure which depicts the 3-dimensional face defined by $H_{12}$ :




*the upper half* *the lower half*

Moreover, one can check (after some efforts) that the kernel is actually generated by this vector together with its translations under the action of the automoprhism group of the 24-cell.

### 3.9.3. The Lowering Operator at Level Three:

This is represented by a $96 \times 24$ -matrix $M_3$ which is the transpose of $M_1$ since the 24-cell $\delta_4$ is self-dual. Hence the rank of $M_3$ is equal to 24, as is seen in 3.9.1, and $\ker M_3 = \{0\}$.

Thus we have completed the proof of Theorem 3.1.

**Remark.** The *only* reason why we exclude the 600-cell and its dual 120-cell is that the matrices of the lowering operators for them are too huge to handle. The author, however, suspects that the kernels at any level are generated by $\pm 1$ -vectors in these cases too.

## 4. Abelian Varieties of CM-Type Associated to Ranked Posets

In this section, we show that one can associate an abelian variety to a ranked poset under a certain condition which is satisfied by every poset

investigated in the previous section. Thereafter, we investigate the structure of the ring of Hodge cycles using the notion of *N-dominatedness* introduced in [4].

### 4.0. Preliminaries

For convenience to the reader, we recall the definition of $N$-dominatedness. For an abelian variety $A$ over $\mathbf{C}$, let $B^*(A) = \bigoplus_{0 \le p \le \dim A} (H^{2p}(A, \mathbf{Q}) \cap H^{p,p}(A))$ denote the ring of Hodge cycles. A Hodge cycle on abelian variety $A$ is said to be *primitive* if it does not belong to the ideal of $B^*(A)$ generated by the divisor classes.

**Definition 4.0.1.** Let $N$ be a positive integer. Given an absolutely simple abelian variety $A$ of CM-type, suppose that for any primitive Hodge cycle $Z$ on any power $A^k$, there exist a positive integer $\ell$ and an element $D \in B^*((A^k)^\ell)$, which is an intersection (possibly empty) of divisor classes, such that $\pi_1^*(Z).D$ is nontrivial and belongs to the subring of $B^*((A^k)^\ell)$ generated by $\bigoplus_{1 \le i \le N} B^i((A^k)^\ell)$, where $\pi_1 : \underbrace{A^k \times \cdots \times A^k}_{\ell \, times} \to A^k$ denotes the projection onto the first copy of $A^k$. Then we say $A$ is *N-dominated*. We denote the minimum of such an $N$ by $d(A)$, and call it *the index of domination of A*.

We will determine the index of domination of each of abelian varieties associated to the face posets of regular polytopes of any dimension.

First, we show the following proposition which is easy to prove but plays a fundamental role for our construction carried out in this section.

**Proposition 4.0.2.** *Suppose that a ranked poset* $P = \coprod_{0 \le i \le n} P_i$ *satisfies the following conditions*:

(i) *A finite group G acts on* $P_i$ *and* $P_{i-1}$, *and the induced actions on* $V(P_i)$ *and* $V(P_{i-1})$ *are compatible with the raising operator*

$U_{i-1} : V(P_{i-1}) \to V(P_i)$.

(ii) *There exist totally real fields* $K_i$, $K_{i-1}$ *with common Galois closure* $F$ *such that* $Gal(F/\mathbf{Q}) \cong G$ *and that* $P_j \cong \mathrm{Hom}\,(K_j, \mathbf{C})$, $j = i-1, i$, *as right G-sets.*

*Then there exists an abelian variety* $A$ *of CM-type such that* $End^0 A \cong K_i(\sqrt{-1})$ *and the analytic representation of* $G \times \{id, \rho\}$ $(\cong Gal\,(F(\sqrt{-1})/\mathbf{Q}\,))$ *given by its action on the space of holomorphic differential forms of* $A$ *is equivalent to the one defined by* $U_{i-1}(p)\coprod \rho(P_i - U_{i-1}(p))$ $(\subset P_i)$ *for some* $p \in P_{i-1}$.

**Proof of Proposition 4.0.2.** In view of [10], it suffices to show that $S = U_{i-1}(p)\coprod \rho(P_i - U_{i-1}(p))$ is a CM-type in the sense that $S \cup \rho S = G \times \{id, \rho\}$ and that $S \cap \rho S = \varnothing$. But both of these equalities are obvious.

Now, we consider what kind of abelian varieties arise from this proposition for the posets investigated in the previous section.

### 4.1. Case of the Regular Simplex $\alpha_n$

Let $f(x) \in \mathbf{Q}[x]$ be an irreducible polynomial of degree $n+1$ with roots $z_1, \ldots, z_{n+1} \in \mathbf{C}$ such that the field $F = \mathbf{Q}(z_1, \ldots, z_{n+1})$ is a totally real extension of $\mathbf{Q}$ with $Gal(F/\mathbf{Q}) \cong S_{n+1}$. (See [11] for the existence of such an irreducible polynomial.) For every $i$ with $1 \le i \le n$, let $y_i = \sum_{1\le k\le i} z_k \in F$ and let $K_i = \mathbf{Q}(y_i) \subset F$. Then the set of all of its conjugates is given by $\left\{\sum_{1\le k\le i} z_{\ell_k} ; 1 \le \ell_1 < \cdots < \ell_i \le n+1\right\}$, and hence is naturally identified with the rank-$i$ part $P(\alpha_n)_i$ of the ranked poset $P(\alpha_n)$. It is obvious that the raising operators $U_{i-1} : V(P(\alpha_n)_{i-1}) \to V(P(\alpha_n)_i)$, $1 \le i \le n$, are compatible with the action of the symmetric

group $S_{n+1}$. Moreover, since there exists no element in $S_{n+1}$ which fixes all the $i$-dimensional faces of $\alpha_n$ other than the identity, the Galois closure of $K_i = \mathbf{Q}(y_i)$ coincides with $F$ for $1 \le i \le n$. Thus the poset $P(\alpha_n)$ satisfies the conditions (i) and (ii) in Proposition 4.02 for any $i$, hence there exists an abelian variety $A(\alpha_n)_i$ of CM-type of dimension $\binom{n+1}{i+1}$ with the properties asserted in the proposition. Furthermore, it follows from [4] that each element of the kernel of the lowering operator $D_i$ gives rise to a Hodge cycle on some power $(A(\alpha_n)_i)^\ell$, $\ell \ge 1$, of the abelian variety $A(\alpha_n)_i$, and, in particular, that a $\pm 1$-vector of height $h$ corresponds to a Hodge cycle of codimension $h$ on $A(\alpha_n)_i$ itself. Hence the index of domination of $A(\alpha_n)_i$ coincides with the dominating height at level $i$ of the poset $P(\alpha_n)$.

Therefore we obtain the following:

**Theorem 4.1.1.** *For any n and i with* $1 \le i \le n/2$, *the abelian variety* $A(\alpha_n)_i$ *is of dimension* $\binom{n+1}{i+1}$ *and 2-dominated. In particular, if the Hodge conjecture holds in codimension two for* $A(\alpha_n)_i$, *then the whole Hodge conjecture holds for every power* $(A(\alpha_n)_i)^\ell$, $\ell \ge 1$.

### 4.2. Case of the Measure Polytope $\gamma_n$

Let $F$ be a totally real extension of $\mathbf{Q}$ with $Gal(F/\mathbf{Q}) \cong (\mathbf{Z}/2\mathbf{Z}) \operatorname{wr} S_n$, which is the Weyl group of type $B_n$. (See [11] for the existence of such a field.) By a similar reasoning to the one given for the case of the regular simplex, Proposition 4.02 enables us to associate an abelian variety $A(\gamma_n)_i$ of CM-type of dimension $2^{n-i}\binom{n}{i}$ to the poset $P(\gamma_n)$ for each $i$. Thus we obtain the following:

**Theorem 4.2.1.** *For any n and i with $1 \le i \le n/2$, the abelian variety $A(\gamma_n)_i$ is of dimension $2^{n-i}\binom{n}{i}$ and $2^{i-1}$-dominated. In particular, if the Hodge conjecture holds in codimension $2^{i-1}$ for $A(\gamma_n)_i$, then the whole Hodge conjecture holds for every power $(A(\gamma_n)_i)^\ell$, $\ell \ge 1$.*

### 4.3. Case of the Cross Polytope $\beta_n$

Since we can use the same totally real field as in the subsection 4.2, a similar argument to the one given above enables us to obtain the following:

**Theorem 4.3.1.** *For any n and i with $1 \le i \le n/2$, the abelian variety $A(\beta_n)_i$ is of dimension $2^{i+1}\binom{n}{i+1}$ and $2^{2i-1}$-dominated. In particular, if the Hodge conjecture holds in codimension $2^{2i-1}$ for $A(\beta_n)_i$, then the whole Hodge conjecture holds for every power $(A(\beta_n)_i)^\ell$, $\ell \ge 1$.*

### 4.4. Case of the Regular Polygons $\pi_n$

It is easily seen that there exists a totally real extension $F$ of $\mathbf{Q}$ with $Gal(F/\mathbf{Q}) \cong \mathbf{Z}/n\mathbf{Z}$. In the set of embeddings $\mathrm{Hom}(F(\sqrt{-1}), \mathbf{C}) \cong \mathbf{Z}/n\mathbf{Z} \times \{id, \rho\}$, we can take the subset $S = \{(0, id), (1, id), (2, \rho), \ldots, (n-1, \rho)\}$ as a CM-type which corresponds to the lowering operator $D_1 : V(P(\pi)_1) \to V(P(\pi)_0)$. Hence we can define an abelian variety $A(\pi_n)$ of dimension $n$ by Proposition 4.02, and by Theorem 3.1 we obtain the following:

**Theorem 4.4.1.** *If $n$ is odd, then the abelian variety $A(\pi_n)$ is nondegenerate. If $n$ is even, then the abelian variety $A(\pi_n)$ is $n/2$-dominated.*

### 4.5. Cases of Tetrahedron, Cube, Octahedron

These are special cases of the regular polytopes treated in the previous subsections.

### 4.6. Case of Dodecahedron $\mathcal{D}$

Let $F$ be a totally real extension of $\mathbf{Q}$ with $Gal(F/\mathbf{Q})$ isomorphic to the icosahedral group $\mathcal{I}_3$, which is of order 120. (The existence of such a field follows from that of a totally real extension with Galois group $A_5$ [3].) Then by a similar method to the one employed in subsection 4.1, we can associate an abelian variety $A(\mathcal{D})$ of CM-type, which is of dimension 30, to the rank-one-part $P(\mathcal{D})_1$ of the face poset of the dodecahedron. Thus by Theorem 3.1 we obtain the following:

**Theorem 4.6.1.** *The* 30-*dimensional abelian variety* $A(\mathcal{D})$ *of CM-type associated to* $P(\mathcal{D})_1$ *is* 4-*dominated. In particular, if the Hodge conjecture holds in codimension four for* $A(\mathcal{D})$, *then the whole Hodge conjecture holds for every power* $A(\mathcal{D})^\ell$, $\ell \geq 1$.

### 4.7. Case of Icosahedron $\mathcal{I}$

By the same token as in the previous subsection, we obtain the following:

**Theorem 4.7.1.** *The* 30-*dimensional abelian variety* $A(\mathcal{I})$ *of CM-type associated to* $P(\mathcal{I})_1$ *is* 2-*dominated. In particular, if the Hodge conjecture holds in codimension two for* $A(\mathcal{I})$, *then the whole Hodge conjecture holds for every power* $A(\mathcal{I})^\ell$, $\ell \geq 1$.

### 4.8. Cases of 5-cell $\alpha_4$, 16-cell $\beta_4$, and Tessaract $\gamma_4$

These are the special cases of those treated in the subsections 4.1, 4.3, and 4.2, respectively.

### 4.9. Case of 24-cell {3, 4, 3}

It follows from [1] that the commutator subgroup of the automorphism group $G$ of the 24-cell is of index four and that it is an extension of the product of two tetrahedral group $\cong A_4$ by a central subgroup of order two. In particular, $G$ is solvable. Hence it follows from [2] that there exists a totally real extension $F$ of $\mathbf{Q}$ such that $Gal(F/\mathbf{Q}) \cong G$. Therefore, by a similar argument to the one given above, we can associate a 96-dimensional abelian variety $A(\{3, 4, 3\})$ to the rank-one-part $P(\{3, 4, 3\})_1$ of the poset $P(\{3, 4, 3\})$, and we have the following:

**Theorem 4.9.1.** *The 48-dimensional abelian variety $A(\{3, 4, 3\})$ of CM-type is 2-dominated. In particular, if the Hodge conjecture holds in codimension two for $A(\{3, 4, 3\})$, then the whole Hodge conjecture holds for every power $A(\{3, 4, 3\})^{\ell}$, $\ell \geq 1$.*

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# FP STRONGLY CONTINUITY, FP θ-CONTINUITY AND FP WEAKLY θ-CONTINUITY

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## Abstract

We introduce FP strongly continuity, FP θ-continuity and FP weakly θ-continuity in fuzzy bitopological spaces in view of the definition of Sostak and study some of their properties.

## 1. Introduction and Preliminaries

A. P. Sostak [15] defined the fuzzy topology as an extension of Chang's fuzzy topology [5]. S. S. Kumar [11] introduced the notions of fuzzy θ-cluster points and fuzzy δ-cluster points in fuzzy bitopological spaces in the sense of Chang's fuzzy topology. Y. C. Kim [9] introduced r-δ-cluster (r-θ-cluster) points and δ-closure (θ-closure) operators in fuzzy bitopological spaces in view of the definition of Sostak. It is a good extension of the notions of S. S. Kumar.

In this paper, we introduce FP strongly continuity, FP θ-continuity and FP weakly θ-continuity in fuzzy bitopological spaces in view of the

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definition of Sostak. In general, FP strongly continuity $\Rightarrow$ FP continuity $\Rightarrow$ FP $\theta$-continuity $\Rightarrow$ FP weakly $\theta$-continuity. But the converses need not be true.

Throughout this paper, let $X$ be a nonempty set, $I = [0, 1]$ and $I_0 = (0, 1]$. For $\alpha \in I$, $\overline{\alpha}(x) = \alpha$ for all $x \in X$. Let $Pt(X)$ be the family of all fuzzy points in $X$ and the indices $i, j \in \{1, 2\}$ and $i \neq j$.

**Definition 1.1** [15]. A function $\tau : I^X \to I$ is called a *fuzzy topology* on $X$ if it satisfies the following conditions:

(O1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,

(O2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$ for any $\mu_1, \mu_2 \in I^X$,

(O3) $\tau(\bigvee_{k\in\Gamma} \mu_k) \geq \bigwedge_{k\in\Gamma} \tau(\mu_k)$ for any $\{\mu_k\}_{k\in\Gamma} \subset I^X$.

The pair $(X, \tau)$ is called a *fuzzy topological space*.

The triple $(X, \tau_1, \tau_2)$ is called a *fuzzy bitopological space* (fbts, in short) where $\tau_1$ and $\tau_2$ are fuzzy topologies on $X$.

**Definition 1.2** [14]. A function $C : I^X \times I_0 \to I^X$ is called a *fuzzy closure operator* on $X$ if for each $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

(C1) $C(\overline{0}, r) = \overline{0}$,

(C2) $\lambda \leq C(\lambda, r)$,

(C3) $C(\lambda, r) \vee C(\mu, r) = C(\lambda \vee \mu, r)$,

(C4) $C(\lambda, r) \leq C(\lambda, s)$, if $r \leq s$.

The pair $(X, C)$ is called a *fuzzy closure space*.

A fuzzy closure space $(X, C)$ is *topological* if for $\lambda \in I^X$ and $r \in I_0$,

(C5) $C(C(\lambda, r), r) = C(\lambda, r)$.

**Theorem 1.3** [14]. *Let* $(X, \tau_1, \tau_2)$ *be a fbts. For each* $r \in I_0$, $\lambda \in I^X$, *we define an operator* $C_{\tau_i} : I^X \times I_0 \to I^X$ *as follows*:

$$C_{\tau_i}(\lambda, r) = \bigwedge\{\mu \mid \lambda \leq \mu, \tau_i(\overline{1} - \mu) \geq r\}.$$

*Then* $(X, C_{\tau_i})$ *is a topological fuzzy closure space.*

**Theorem 1.4** [9]. *Let* $(X, \tau_1, \tau_2)$ *be a fbts. For each* $r \in I_0$, $\lambda \in I^X$, *we define an operator* $I_{\tau_i} : I^X \times I_0 \to I^X$ *as follows*:

$$I_{\tau_i}(\lambda, r) = \bigvee\{\mu \mid \mu \leq \lambda, \tau_i(\mu) \geq r\}.$$

*For each* $\lambda, \mu \in I^X$, $r, s \in I_0$, *it has the following properties*:

(1) $I_{\tau_i}(\overline{1} - \lambda, r) = \overline{1} - C_{\tau_i}(\lambda, r)$.

(2) *If* $I_{\tau_i}(C_{\tau_j}(\lambda, r), r) = \lambda$, *then* $C_{\tau_i}(I_{\tau_j}(\overline{1} - \lambda, r), r) = \overline{1} - \lambda$.

(3) $I_{\tau_i}(\overline{1}, r) = \overline{1}$.

(4) $I_{\tau_i}(\lambda, r) \leq \lambda$.

(5) $I_{\tau_i}(\lambda, r) \wedge I_{\tau_i}(\mu, r) = I_{\tau_i}(\lambda \wedge \mu, r)$.

(6) $I_{\tau_i}(\lambda, r) \geq I_{\tau_i}(\lambda, s)$, *if* $r \leq s$.

(7) $I_{\tau_i}(I_{\tau_i}(\lambda, r), r) = I_{\tau_i}(\lambda, r)$.

**Definition 1.5** [6, 9]. Let $(X, \tau_1, \tau_2)$ be a fbts, $\mu \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$. Then $\mu$ is called a *r-open* $\mathcal{Q}_{\tau_i}$*-neighborhood* of $x_t$ if $x_t \, q \, \mu$ with $\tau_i(\mu) \geq r$. We denote $\mathcal{Q}_{\tau_i}(x_t, r) = \{\mu \in I^X \mid x_t \, q \, \mu, \tau_i(\mu) \geq r\}$.

**Definition 1.6** [9]. Let $(X, \tau_1, \tau_2)$ be a fbts, $\lambda \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$. Then

(1) $x_t$ is called a $r$-$\tau_i$ *cluster point* of $\lambda$, if for every $\mu \in \mathcal{Q}_{\tau_i}(x_t, r)$, we have $\mu \, q \, \lambda$.

(2) $x_t$ is called a $r$-$(\tau_i, \tau_j)$ $\theta$-*cluster point* of $\lambda$, if for every $\mu \in \mathcal{Q}_{\tau_i}(x_t, r)$, we have $C_{\tau_j}(\mu, r) q \lambda$.

(3) A $(\tau_i, \tau_j)$ $\theta$-*closure operator* is a function $T^{\tau_i}_{\tau_j} : I^X \times I_0 \to I^X$ defined as follows:

$$T^{\tau_i}_{\tau_j}(\lambda, r) = \bigvee \{x_t \in Pt(X) \mid x_t \text{ is a r-}(\tau_i, \tau_j)\ \theta\text{-cluster point of } \lambda\}.$$

**Theorem 1.7** [9]. *Let $(X, \tau_1, \tau_2)$ be a fbts. For each $\lambda, \mu, \rho \in I^X$ and $r \in I_0$, we have the following properties*:

(1) $T^{\tau_i}_{\tau_j}(\lambda, r) = \bigwedge \{\mu \in I^X \mid \lambda \leq I_{\tau_j}(\mu, r),\ \tau_i(\overline{1} - \mu) \geq r\}$.

(2) $C_{\tau_i}(\lambda, r) = \bigvee \{x_t \in Pt(X) \mid x_t$ *is a $r$-$\tau_i$ cluster point of* $\lambda\}$.

(3) $x_t$ *is a $r$-$(\tau_i, \tau_j)$ $\theta$-cluster point of $\lambda$ iff* $x_t \in T^{\tau_i}_{\tau_j}(\lambda, r)$.

(4) $x_t$ *is a $r$-$\tau_i$ cluster point of $\lambda$ iff* $x_t \in C_{\tau_i}(\lambda, r)$.

(5) $\lambda \leq C_{\tau_i}(\lambda, r) \leq T^{\tau_i}_{\tau_j}(\lambda, r)$.

(6) *If* $\tau_j(\lambda) \geq r$, *then* $C_{\tau_i}(\lambda, r) = T^{\tau_i}_{\tau_j}(\lambda, r)$.

(7) $T^{\tau_i}_{\tau_j}(\lambda, r) \leq T^{\tau_i}_{\tau_j}(\mu, r)$, *if* $\lambda \leq \mu$.

## 2. FP Strongly Continuous, FP θ-continuous and FP Weakly θ-continuous Mappings

**Definition 2.1.** Let $(X, \tau_1, \tau_2)$ and $(Y, \eta_1, \eta_2)$ be fbts's. Let $f : X \to Y$ be a function. Then

(1) $f$ is called *FP continuous* iff $\eta_i(\mu) \leq \tau_i(f^{-1}(\mu))$ for each $\mu \in I^Y$.

(2) $f$ is called *FP open* iff $\tau_i(\lambda) \leq \eta_i(f(\lambda))$ for each $\lambda \in I^X$.

(3) $f$ is called *FP closed* iff $\tau_i(\overline{1} - \lambda) \leq \eta_i(\overline{1} - f(\lambda))$ for each $\lambda \in I^X$.

(4) $f$ is called *FP strongly continuous* iff for each $\mu \in \mathcal{Q}_{\eta_i}(f(x)_t, r)$, there exists $\lambda \in \mathcal{Q}_{\tau_i}(x_t, r)$ such that $f(C_{\tau_j}(\lambda, r)) \leq \mu$.

(5) $f$ is called *FP θ-continuous* iff for each $\mu \in \mathcal{Q}_{\eta_i}(f(x)_t, r)$, there exists $\lambda \in \mathcal{Q}_{\tau_i}(x_t, r)$ such that $f(C_{\tau_j}(\lambda, r)) \leq C_{\eta_i}(\mu, r)$.

(6) $f$ is called *FP weakly θ-continuous* iff for each $\mu \in \mathcal{Q}_{\eta_i}(f(x)_t, r)$, there exists $\lambda \in \mathcal{Q}_{\tau_i}(x_t, r)$ such that $f(\lambda) \leq C_{\eta_j}(\mu, r)$.

**Theorem 2.2.** *Let $(X, \tau_1, \tau_2)$ and $(Y, \eta_1, \eta_2)$ be fbts's. Let $f : X \to Y$ be a function. For each $\lambda \in I^X$, $\mu \in I^Y$ and $r \in I_0$, the following statements are equivalent:*

(1) *A map $f$ is FP strongly continuous.*

(2) $f\left(T_{\tau_j}^{\tau_i}(\lambda, r)\right) \leq C_{\eta_i}(f(\lambda), r)$.

(3) $T_{\tau_j}^{\tau_i}(f^{-1}(\mu), r) \leq f^{-1}(C_{\eta_i}(\mu, r))$.

(4) $T_{\tau_j}^{\tau_i}(f^{-1}(\mu), r) = f^{-1}(\mu)$, *for each set* $\mu = C_{\eta_i}(\mu, r)$.

(5) $T_{\tau_j}^{\tau_i}(\overline{1} - f^{-1}(\mu), r) = \overline{1} - f^{-1}(\mu)$, *for each set* $\mu = I_{\eta_i}(\mu, r)$.

**Proof.** (1) ⇒ (2) Suppose there exist $\lambda \in I^X$, $r \in I_0$ and $i, j \in \{1, 2\}$ with $i \neq j$ such that

$$f\left(T_{\tau_j}^{\tau_i}(\lambda, r)\right) \not\leq C_{\eta_i}(f(\lambda), r).$$

Then there exist $y \in Y$ and $t \in I_0$ such that

$$f\left(T_{\tau_j}^{\tau_i}(\lambda, r)\right)(y) > t > C_{\eta_i}(f(\lambda), r)(y).$$

Since $f^{-1}(\{y\}) = \varnothing$, it is a contradiction that $f\left(T_{\tau_j}^{\tau_i}(\lambda, r)\right)(y) = 0$,

$f^{-1}(\{y\}) \neq \varnothing$, hence there exists $x \in f^{-1}(\{y\})$ such that

$$f\Big(T^{\tau_i}_{\tau_j}(\lambda, r)\Big)(y) \geq T^{\tau_i}_{\tau_j}(\lambda, r)(x) > t > C_{\eta_i}(f(\lambda), r)(f(x)). \tag{A}$$

Since $C_{\eta_i}(f(\lambda), r)(f(x)) < t$, by Theorem 1.7(4), $f(x)_t$ is not a r-$\eta_i$ cluster point of $f(\lambda)$. Then there exists $\mu \in \mathcal{Q}_{\eta_i}(f(x)_t, r)$ such that $f(\lambda) \leq \overline{1} - \mu$. Since $f$ is FP strongly continuous, for $\mu \in \mathcal{Q}_{\eta_i}(f(x)_t, r)$, there exists $\nu \in \mathcal{Q}_{\tau_i}(x_t, r)$ such that $f(C_{\tau_j}(\nu, r)) \leq \mu$. Thus, $f(\lambda) \leq \overline{1} - f(C_{\tau_j}(\nu, r))$ implies $\lambda \leq \overline{1} - C_{\tau_j}(\nu, r) = I_{\tau_j}(\overline{1} - \nu, r)$. Since $\tau_i(\nu) \geq r$, by Theorem 1.7(1)

$$T^{\tau_i}_{\tau_j}(\lambda, r) \leq \overline{1} - \nu.$$

Since $x_t \, q \, \nu$, we have $T^{\tau_i}_{\tau_j}(\lambda, r)(x) \leq (\overline{1} - \nu)(x) < t$. It is a contradiction for equation (A).

(2) $\Rightarrow$ (3) For all $\mu \in I^Y$, $r \in I_0$, put $\lambda = f^{-1}(\mu)$ from (2). Then

$$f\Big(T^{\tau_i}_{\tau_j}(f^{-1}(\mu), r)\Big) \leq C_{\eta_i}(f(f^{-1}(\mu)), r) \leq C_{\eta_i}(\mu, r).$$

It implies

$$T^{\tau_i}_{\tau_j}(f^{-1}(\mu), r) \leq f^{-1}\Big(f\Big(T^{\tau_i}_{\tau_j}(f^{-1}(\mu), r)\Big)\Big) \leq f^{-1}(C_{\eta_i}(\mu, r)).$$

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are easily proved from Theorem 1.7(5) and Theorem 1.4(1).

(5) $\Rightarrow$ (1) Let $\mu \in \mathcal{Q}_{\eta_i}(f(x)_t, r)$. Since $\mu = I_{\eta_i}(\mu, r)$, by (5)

$$\overline{1} - f^{-1}(\mu) = T^{\tau_i}_{\tau_j}(\overline{1} - f^{-1}(\mu), r).$$

Since $f(x)_t \, q \, \mu$, we have $x_t \, q \, f^{-1}(\mu)$, that is,

$$t > (\overline{1} - f^{-1}(\mu))(x) = T^{\tau_i}_{\tau_j}(\overline{1} - f^{-1}(\mu), r)(x).$$

Thus, $x_t$ is not a r-$(\tau_i, \tau_j)$ θ-cluster point of $\overline{1} - f^{-1}(\mu)$. Then there exists $\nu \in Q_{\tau_i}(x_t, r)$ such that

$$\overline{1} - f^{-1}(\mu) \leq \overline{1} - C_{\tau_j}(\nu, r).$$

Hence $C_{\tau_j}(\nu, r) \leq f^{-1}(\mu)$ implies $f(C_{\tau_j}(\nu, r)) \leq \mu$.

The following theorem is similarly proved as Theorem 2.2.

**Theorem 2.3.** *Let* $(X, \tau_1, \tau_2)$ *and* $(Y, \eta_1, \eta_2)$ *be fbts's. Let* $f : X \to Y$ *be a function. For each* $\lambda \in I^X$, $\mu \in I^Y$ *and* $r \in I_0$, *the following statements are equivalent:*

(1) *A map f is FP continuous.*

(2) $f(C_{\tau_i}(\lambda, r)) \leq C_{\eta_i}(f(\lambda), r)$.

(3) $C_{\tau_i}(f^{-1}(\mu), r) \leq f^{-1}(C_{\eta_i}(\mu, r))$.

**Theorem 2.4.** *Let* $(X, \tau_1, \tau_2)$ *and* $(Y, \eta_1, \eta_2)$ *be fbts's. Let* $f : X \to Y$ *be a function. For each* $\lambda \in I^X$, $\mu \in I^Y$ *and* $r \in I_0$, *the following statements are equivalent:*

(1) *A map f is FP θ-continuous.*

(2) $f\left(T^{\tau_i}_{\tau_j}(\lambda, r)\right) \leq T^{\eta_i}_{\eta_j}(f(\lambda), r)$.

(3) $T^{\tau_i}_{\tau_j}(f^{-1}(\mu), r) \leq f^{-1}\left(T^{\eta_i}_{\eta_j}(\mu, r)\right)$.

**Proof.** (1) ⇒ (2) Suppose there exist $\lambda \in I^X$, $r \in I_0$ and $i, j \in \{1, 2\}$ with $i \neq j$ such that

$$f\left(T^{\tau_i}_{\tau_j}(\lambda, r)\right) \not\leq T^{\eta_i}_{\eta_j}(f(\lambda), r).$$

Then there exist $y \in Y$ and $t \in I_0$ such that

$$f\left(T^{\tau_i}_{\tau_j}(\lambda, r)\right)(y) > t > T^{\eta_i}_{\eta_j}(f(\lambda), r)(y).$$

Since $f^{-1}(\{y\}) = \emptyset$, provides a contradiction that $f\left(T_{\tau_j}^{\tau_i}(\lambda, r)\right)(y) = 0$, $f^{-1}(\{y\}) \neq \emptyset$, and hence there exists $x \in f^{-1}(\{y\})$ such that

$$f\left(T_{\tau_j}^{\tau_i}(\lambda, r)\right)(y) \geq T_{\tau_j}^{\tau_i}(\lambda, r)(x) > t > T_{\eta_j}^{\eta_i}(f(\lambda), r)(f(x)). \tag{B}$$

Since $T_{\eta_j}^{\eta_i}(f(\lambda), r)(f(x)) < t$, by Theorem 1.7(3), $f(x)_t$ is not a r-$(\eta_i, \eta_j)$ $\theta$-cluster point of $f(\lambda)$. Then there exists $\mu \in \mathcal{Q}_{\eta_i}(f(x)_t, r)$ such that $f(\lambda) \leq \overline{1} - C_{\eta_j}(\mu, r)$. Since $f$ is FP $\theta$-continuous, for $\mu \in \mathcal{Q}_{\eta_i}(f(x)_t, r)$, there exists $\nu \in \mathcal{Q}_{\tau_i}(x_t, r)$ such that $f(C_{\tau_j}(\nu, r)) \leq C_{\eta_j}(\mu, r)$. Thus, $f(\lambda) \leq \overline{1} - f(C_{\tau_j}(\nu, r))$ implies $\lambda \leq \overline{1} - C_{\tau_j}(\nu, r)$. Hence $x_t$ is not a r-$(\tau_i, \tau_j)$ $\theta$-cluster point of $\lambda$, by Theorem 1.7(3), $T_{\tau_j}^{\tau_i}(\lambda, r)(x) < t$. It is a contradiction for equation (B).

(2) $\Rightarrow$ (3) For all $\mu \in I^Y$, $r \in I_0$, put $\lambda = f^{-1}(\mu)$ from (2). Then

$$f\left(T_{\tau_j}^{\tau_i}(f^{-1}(\mu), r)\right) \leq T_{\eta_j}^{\eta_i}(f(f^{-1}(\mu)), r) \leq T_{\eta_j}^{\eta_i}(\mu, r).$$

It implies

$$T_{\tau_j}^{\tau_i}(f^{-1}(\mu), r) \leq f^{-1}\left(f\left(T_{\tau_j}^{\tau_i}(f^{-1}(\mu), r)\right)\right) \leq f^{-1}\left(T_{\eta_j}^{\eta_i}(\mu, r)\right).$$

(3) $\Rightarrow$ (1) Let $\mu \in \mathcal{Q}_{\eta_i}(f(x)_t, r)$. Since $(\overline{1} - \mu)(f(x)) < t$ and

$$T_{\eta_j}^{\eta_i}(\overline{1} - C_{\eta_j}(\mu, r), r) = T_{\eta_j}^{\eta_i}(I_{\eta_j}(\overline{1} - \mu, r), r)$$

$$\leq \overline{1} - \mu, \quad \text{(by Theorem 1.7(1))}$$

we have

$$T_{\eta_j}^{\eta_i}(\overline{1} - C_{\eta_j}(\mu, r), r)(f(x)) \leq (\overline{1} - \mu)(f(x)) < t.$$

From (3), it implies

$$t > T_{\eta_j}^{\eta_i}(\overline{1} - C_{\eta_j}(\mu, r), r)(f(x))$$

$$= f^{-1}\Big(T^{\eta_i}_{\eta_j}(\overline{1} - C_{\eta_j}(\mu, r), r)\Big)(x)$$

$$\geq T^{\tau_i}_{\tau_j}(f^{-1}(\overline{1} - C_{\eta_j}(\mu, r)), r)(x).$$

Thus, $x_t$ is not a r-$(\tau_i, \tau_j)$ θ-cluster point of $f^{-1}(\overline{1} - C_{\eta_j}(\mu, r))$. Then there exists $\nu \in \mathcal{Q}_{\tau_i}(x_t, r)$ such that

$$f^{-1}(\overline{1} - C_{\eta_j}(\mu, r)) \leq \overline{1} - C_{\tau_j}(\nu, r).$$

Hence $C_{\tau_j}(\nu, r) \leq f^{-1}(C_{\eta_j}(\mu, r))$ implies $f(C_{\tau_j}(\nu, r)) \leq C_{\eta_j}(\mu, r)$.

**Theorem 2.5.** *Let* $(X, \tau_1, \tau_2)$ *and* $(Y, \eta_1, \eta_2)$ *be fbts's. Let* $f : X \to Y$ *be a function. For each* $\lambda \in I^X$, $\mu \in I^Y$ *and* $r \in I_0$, *the following statements are equivalent*:

(1) *A map f is FP weakly* θ*-continuous.*

(2) $f(C_{\tau_i}(\lambda, r)) \leq T^{\eta_i}_{\eta_j}(f(\lambda), r)$.

(3) $C_{\tau_i}(f^{-1}(\mu), r) \leq f^{-1}(T^{\eta_i}_{\eta_j}(\mu, r))$.

(4) $f^{-1}(\mu) \leq I_{\tau_i}(f^{-1}(C_{\eta_j}(\mu, r)), r)$, *for each* $\eta_i(\mu) \geq r$.

**Proof.** (1) ⇔ (2) ⇔ (3) It is similarly proved as Theorem 2.4.

(1) ⇒ (4) Let $x_t \, q \, f^{-1}(\mu)$ and $\eta_i(\mu) \geq r$. Then $\mu \in \mathcal{Q}_{\eta_i}(f(x)_t, r)$. Since $f$ is FP weakly θ-continuous, there exists $\nu \in \mathcal{Q}_{\tau_i}(x_t, r)$ such that

$$f(\nu) \leq C_{\eta_j}(\mu, r).$$

So, $\nu \leq f^{-1}(C_{\eta_j}(\mu, r))$ implies $\nu = I_{\tau_i}(\nu, r) \leq I_{\tau_i}(f^{-1}(C_{\eta_j}(\mu, r)), r)$. Thus, $x_t \, q \, f^{-1}(\mu)$ implies $x_t \, q \, I_{\tau_i}(f^{-1}(C_{\eta_j}(\mu, r)), r)$. Hence

$$f^{-1}(\mu) \leq I_{\tau_i}(f^{-1}(C_{\eta_j}(\mu, r)), r).$$

(4) $\Rightarrow$ (1) Let $\mu \in \mathcal{Q}_{\eta_i}(f(x)_t, r)$. Since $f^{-1}(\mu) \leq I_{\tau_i}(f^{-1}(C_{\eta_j}(\mu, r)), r)$ from (4), we have

$$x_t \, q \, I_{\tau_i}(f^{-1}(C_{\eta_j}(\mu, r)), r), \quad \tau_i(I_{\tau_i}(f^{-1}(C_{\eta_j}(\mu, r)), r)) \geq r.$$

Thus, $I_{\tau_i}(f^{-1}(C_{\eta_j}(\mu, r)), r) \in \mathcal{Q}_{\tau_i}(x_t, r)$. Moreover,

$$\begin{aligned} f(I_{\tau_i}(f^{-1}(C_{\eta_j}(\mu, r)), r)) &\leq f(f^{-1}(C_{\eta_j}(\mu, r))) \\ &\leq C_{\eta_j}(\mu, r). \end{aligned}$$

**Theorem 2.6.** *We have the following implications*:

*FP strongly continuity* $\Rightarrow$ *FP continuity*

$\Rightarrow$ *FP* $\theta$-*continuity* $\Rightarrow$ *FP weakly* $\theta$-*continuity.*

**Proof.** We only show FP continuity implies FP $\theta$-continuity because others are easily proved from Definition 2.1. Let $(X, \tau_1, \tau_2)$ and $(Y, \eta_1, \eta_2)$ be fbts's. Let $f : X \to Y$ be FP continuous. From Theorem 2.4(2), suppose there exist $\lambda \in I^X$, $r \in I_0$ and $i, j \in \{1, 2\}$ with $i \neq j$ such that

$$f\left(T^{\tau_i}_{\tau_j}(\lambda, r)\right) \not\leq T^{\eta_i}_{\eta_j}(f(\lambda), r).$$

Then there exist $y \in Y$ and $t \in I_0$ such that

$$f\left(T^{\tau_i}_{\tau_j}(\lambda, r)\right)(y) > t > T^{\eta_i}_{\eta_j}(f(\lambda), r)(y).$$

Since $f^{-1}(\{y\}) = \varnothing$, provides a contradiction that $f\left(T^{\tau_i}_{\tau_j}(\lambda, r)\right)(y) = 0$, $f^{-1}(\{y\}) \neq \varnothing$, and hence there exists $x \in f^{-1}(\{y\})$ such that

$$f\left(T^{\tau_i}_{\tau_j}(\lambda, r)\right)(y) \geq T^{\tau_i}_{\tau_j}(\lambda, r)(x) > t > T^{\eta_i}_{\eta_j}(f(\lambda), r)(f(x)). \tag{C}$$

Since $T^{\eta_i}_{\eta_j}(f(\lambda), r)(f(x)) < t$, by Theorem 1.7(3), $f(x)_t$ is not a r-$(\eta_i, \eta_j)$ $\theta$-cluster point of $f(\lambda)$. Then there exists $\mu \in \mathcal{Q}_{\eta_i}(f(x)_t, r)$ such that

$f(\lambda) \leq \overline{1} - C_{\eta_j}(\mu, r)$. Since $f$ is FP continuous, for $\mu \in \mathcal{Q}_{\eta_i}(f(x)_t, r)$, there exists $f^{-1}(\mu) \in \mathcal{Q}_{\tau_i}(x_t, r)$. Then $f(\lambda) \leq \overline{1} - C_{\eta_j}(\mu, r)$ implies

$$\lambda \leq \overline{1} - f^{-1}(C_{\eta_j}(\mu, r))$$

$$\leq \overline{1} - C_{\tau_j}(f^{-1}(\mu), r) \text{ (by Theorem 2.3(3))}$$

$$= I_{\tau_j}(\overline{1} - f^{-1}(\mu), r).$$

Thus, by Theorem 1.7(1),

$$T_{\tau_j}^{\tau_i}(\lambda, r)(x) \leq (\overline{1} - f^{-1}(\mu))(x) < t.$$

It is a contradiction for equation (C). Thus, $f$ is FP θ-continuous.

**Example 2.7.** Define fuzzy topologies $\sigma_i : I^X \to I$ as follows:

$$\sigma_1(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{2}{3}, & \text{if } \lambda = \overline{0.4}, \\ \frac{2}{3}, & \text{if } \lambda = \overline{0.3}, \\ 0, & \text{otherwise,} \end{cases} \qquad \sigma_2(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{2}{3}, & \text{if } \lambda = \overline{0.3}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\sigma_3(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{2}{3}, & \text{if } \lambda = \overline{0.4}, \\ 0, & \text{otherwise,} \end{cases} \qquad \sigma_4(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{2}{3}, & \text{if } \lambda = \overline{0.5}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\sigma_5(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{2}{3}, & \text{if } \lambda = \overline{0.6}, \\ 0, & \text{otherwise.} \end{cases}$$

(1) Let $\tau_1 = \sigma_1$, $\tau_2 = \sigma_4$, $\eta_1 = \sigma_3$ and $\eta_2 = \sigma_4$. Let $(X, \tau_1, \tau_2)$ and $(X, \eta_1, \eta_2)$ be fbts's. By Theorem 1.7(1), we obtain $T_{\tau_2}^{\tau_1}, T_{\tau_1}^{\tau_2} : I^X \times I_0 \to I^X$ as follows:

$$T_{\tau_2}^{\tau_1}(\lambda, r) = \begin{cases} \overline{0}, & \text{if } \lambda = \overline{0},\ r \in I_0, \\ \overline{0.6}, & \text{if } \overline{0} \neq \lambda \leq \overline{0.5},\ 0 < r \leq \dfrac{2}{3}, \\ \overline{1}, & \text{otherwise,} \end{cases}$$

$$T_{\tau_1}^{\tau_2}(\lambda, r) = \begin{cases} \overline{0}, & \text{if } \lambda = \overline{0},\ r \in I_0, \\ \overline{0.5}, & \text{if } \overline{0} \neq \lambda \leq \overline{0.4},\ 0 < r \leq \dfrac{2}{3}, \\ \overline{1}, & \text{otherwise.} \end{cases}$$

We obtain $C_{\eta_1}, C_{\eta_2} : I^X \times I_0 \to I^X$ as follows:

$$C_{\eta_1}(\lambda, r) = \begin{cases} \overline{0}, & \text{if } \lambda = \overline{0},\ r \in I_0, \\ \overline{0.6}, & \text{if } \overline{0} \neq \lambda \leq \overline{0.6},\ 0 < r \leq \dfrac{2}{3}, \\ \overline{1}, & \text{otherwise,} \end{cases}$$

$$C_{\eta_2}(\lambda, r) = \begin{cases} \overline{0}, & \text{if } \lambda = \overline{0},\ r \in I_0, \\ \overline{0.5}, & \text{if } \overline{0} \neq \lambda \leq \overline{0.5},\ 0 < r \leq \dfrac{2}{3}, \\ \overline{1}, & \text{otherwise.} \end{cases}$$

Since $\tau_i(\lambda) \leq \eta_i(\lambda)$ for each $\lambda \in I^X$, the identity function $id_X : (X, \tau_1, \tau_2) \to (X, \eta_1, \eta_2)$ is FP continuous but not FP strongly continuous because, by Theorem 2.2(2),

$$\overline{1} = T_{\tau_2}^{\tau_1}\left(\overline{0.6}, \frac{2}{3}\right) > C_{\eta_1}\left(\overline{0.6}, \frac{2}{3}\right) = \overline{0.6},$$

$$\overline{1} = T_{\tau_1}^{\tau_2}\left(\overline{0.5}, \frac{2}{3}\right) > C_{\eta_2}\left(\overline{0.5}, \frac{2}{3}\right) = \overline{0.5}.$$

(2) Let $\tau_1 = \sigma_3$, $\tau_2 = \sigma_4$, $\eta_1 = \sigma_2$ and $\eta_2 = \sigma_4$. Let $(X, \tau_1, \tau_2)$ and $(X, \eta_1, \eta_2)$ be fbts's. We obtain $T_{\tau_2}^{\tau_1}, T_{\tau_1}^{\tau_2} : I^X \times I_0 \to I^X$ as follows:

$$T_{\tau_2}^{\tau_1}(\lambda, r) = \begin{cases} \overline{0}, & \text{if } \lambda = \overline{0},\ r \in I_0, \\ \overline{0.6}, & \text{if } \overline{0} \neq \lambda \leq \overline{0.5},\ 0 < r \leq \dfrac{2}{3}, \\ \overline{1}, & \text{otherwise,} \end{cases}$$

$$T_{\tau_1}^{\tau_2}(\lambda, r) = \begin{cases} \overline{0}, & \text{if } \lambda = \overline{0},\ r \in I_0, \\ \overline{0.5}, & \text{if } \overline{0} \neq \lambda \leq \overline{0.4},\ 0 < r \leq \dfrac{2}{3}, \\ \overline{1}, & \text{otherwise.} \end{cases}$$

We obtain $T_{\eta_2}^{\eta_1}, T_{\eta_1}^{\eta_2} : I^X \times I_0 \to I^X$ as follows:

$$T_{\eta_2}^{\eta_1}(\lambda, r) = \begin{cases} \overline{0}, & \text{if } \lambda = \overline{0},\ r \in I_0, \\ \overline{0.7}, & \text{if } \overline{0} \neq \lambda \leq \overline{0.5},\ 0 < r \leq \dfrac{2}{3}, \\ \overline{1}, & \text{otherwise,} \end{cases}$$

$$T_{\eta_1}^{\eta_2}(\lambda, r) = \begin{cases} \overline{0}, & \text{if } \lambda = \overline{0},\ r \in I_0, \\ \overline{0.5}, & \text{if } \overline{0} \neq \lambda \leq \overline{0.3},\ 0 < r \leq \dfrac{2}{3}, \\ \overline{1}, & \text{otherwise.} \end{cases}$$

Then the identity function $id_X : (X, \tau_1, \tau_2) \to (X, \eta_1, \eta_2)$ is FP θ-continuous because $T_{\tau_j}^{\tau_i}(\lambda, r) \leq T_{\eta_j}^{\eta_i}(\lambda, r)$ for each $\lambda \in I^X$ and $r \in I_0$. But it is not FP continuous because

$$\frac{2}{3} = \eta_1(\overline{0.3}) > \tau_1(\overline{0.3}) = 0.$$

(3) Let $\tau_1 = \sigma_4$, $\tau_2 = \sigma_5$, $\eta_1 = \sigma_3$ and $\eta_2 = \sigma_4$. Let $(X, \tau_1, \tau_2)$ and $(X, \eta_1, \eta_2)$ be fbts's. We obtain $C_{\tau_1}, C_{\tau_2} : I^X \times I_0 \to I^X$ as follows:

$$C_{\tau_1}(\lambda, r) = \begin{cases} \overline{0}, & \text{if } \lambda = \overline{0},\ r \in I_0, \\ \overline{0.5}, & \text{if } \overline{0} \neq \lambda \leq \overline{0.5},\ 0 < r \leq \dfrac{2}{3}, \\ \overline{1}, & \text{otherwise,} \end{cases}$$

$$C_{\tau_2}(\lambda, r) = \begin{cases} \overline{0}, & \text{if } \lambda = \overline{0},\ r \in I_0, \\ \overline{0.4}, & \text{if } \overline{0} \neq \lambda \leq \overline{0.4},\ 0 < r \leq \dfrac{2}{3}, \\ \overline{1}, & \text{otherwise.} \end{cases}$$

We obtain $T_{\tau_2}^{\tau_1}, T_{\tau_1}^{\tau_2} : I^X \times I_0 \to I^X$ as follows:

$$T_{\tau_2}^{\tau_1}(\lambda, r) = \begin{cases} \overline{0}, & \text{if } \lambda = \overline{0}, r \in I_0, \\ \overline{1}, & \text{otherwise.} \end{cases}$$

We obtain $T_{\eta_2}^{\eta_1}, T_{\eta_1}^{\eta_2} : I^X \times I_0 \to I^X$ as follows:

$$T_{\eta_2}^{\eta_1}(\lambda, r) = \begin{cases} \overline{0}, & \text{if } \lambda = \overline{0}, r \in I_0, \\ \overline{0.6}, & \text{if } \overline{0} \neq \lambda \leq \overline{0.5},\ 0 < r \leq \dfrac{2}{3}, \\ \overline{1}, & \text{otherwise,} \end{cases}$$

$$T_{\eta_1}^{\eta_2}(\lambda, r) = \begin{cases} \overline{0}, & \text{if } \lambda = \overline{0}, r \in I_0, \\ \overline{0.5}, & \text{if } \overline{0} \neq \lambda \leq \overline{0.4},\ 0 < r \leq \dfrac{2}{3}, \\ \overline{1}, & \text{otherwise.} \end{cases}$$

Then the identity function $id_X : (X, \tau_1, \tau_2) \to (X, \eta_1, \eta_2)$ is FP weakly θ-continuous because $C_{\tau_i}(\lambda, r) \leq T_{\eta_j}^{\eta_i}(\lambda, r)$ for each $\lambda \in I^X$ and $r \in I_0$. But it is not FP θ-continuous from

$$\overline{1} = T_{\tau_2}^{\tau_1}\left(\overline{0.5}, \frac{2}{3}\right) > T_{\eta_2}^{\eta_1}\left(\overline{0.5}, \frac{2}{3}\right) = \overline{0.6},$$

$$\overline{1} = T_{\tau_1}^{\tau_2}\left(\overline{0.4}, \frac{2}{3}\right) > T_{\eta_1}^{\eta_2}\left(\overline{0.4}, \frac{2}{3}\right) = \overline{0.5}.$$

**Theorem 2.8.** *Let* $(X, \tau_1, \tau_2)$ *and* $(Y, \eta_1, \eta_2)$ *be fbts's and* $f : X \to Y$ *be a function.*

(1) *If f is FP strongly continuous and FP-open, then, for each* $\mu \in I^Y$ *and* $r \in I_0$, $T_{\tau_j}^{\tau_i}(f^{-1}(\mu), r) = f^{-1}(C_{\eta_i}(\mu, r))$.

(2) *If f is FP continuous and FP-open, then, for each* $\mu \in I^Y$ *and* $r \in I_0$, $C_{\tau_i}(f^{-1}(\mu), r) = f^{-1}(C_{\eta_i}(\mu, r))$.

**Proof.** (1) From Theorem 2.2(3), we only show that, for each $\mu \in I^Y$ and $r \in I_0$,

$$T_{\tau_j}^{\tau_i}(f^{-1}(\mu), r) \geq f^{-1}(C_{\eta_i}(\mu, r)).$$

Suppose there exist $\mu \in I^Y$, $r \in I_0$ and $i, j \in \{1, 2\}$ with $i \neq j$ such that

$$T_{\tau_j}^{\tau_i}(f^{-1}(\mu), r) \not\geq f^{-1}(C_{\eta_i}(\mu, r)). \tag{D}$$

Then there exist $x \in X$ and $t \in I_0$ such that

$$T_{\tau_j}^{\tau_i}(f^{-1}(\mu), r)(x) < t < C_{\eta_i}(\mu, r)(f(x)).$$

Since $T_{\tau_j}^{\tau_i}(f^{-1}(\mu), r)(x) < t$, by Theorem 1.7(3), $x_t$ is not a r-$(\tau_i, \tau_j)$ θ-cluster point of $f^{-1}(\mu)$. Hence there exists $\rho \in \mathcal{Q}_{\tau_i}(x_t, r)$ such that

$$f^{-1}(\mu) \leq \overline{1} - C_{\tau_j}(\rho, r) \leq \overline{1} - \rho.$$

Furthermore, $x_t \, q \, \rho$ implies $f(x)_t \, q \, f(\rho)$. Since $f$ is FP-open, $\eta_i(f(\rho)) \geq \tau_i(\rho) \geq r$. Thus, $f(\rho) \in \mathcal{Q}_{\eta_i}(f(x)_t, r)$. Since $f^{-1}(\mu) \leq \overline{1} - \rho$ iff $\mu \leq \overline{1} - f(\rho)$, $f(x)_t$ is not a r-$\eta_i$-cluster point of $\mu$, that is, $C_{\eta_i}(\mu, r)(f(x)) < t$. It is a contradiction for equation (D).

(2) It is similar to (1).

## 3. FP Regular Spaces

**Definition 3.1.** A fbts $(X, \tau_1, \tau_2)$ is called *FP regular* iff for each $\mu \in \mathcal{Q}_{\tau_i}(x_t, r)$, there exists $\rho \in \mathcal{Q}_{\tau_i}(x_t, r)$ such that $C_{\tau_j}(\rho, r) \leq \mu$.

**Example 3.2.** Define fuzzy topologies $\sigma_i : I^X \to I$ as in Example 2.7.

(1) Let $\tau_1 = \sigma_3$ and $\tau_2 = \sigma_5$. A fbts $(X, \tau_1, \tau_2)$ is FP regular because for $\overline{0.4} \in \mathcal{Q}_{\tau_1}(x_t, r)$ with $t > 0.6$ and $0 < r \leq \frac{2}{3}$, there exists

$\overline{0.4} \in \mathcal{Q}_{\tau_1}(x_t, r)$ such that $\overline{0.4} = C_{\tau_2}(\overline{0.4}, r)$ and for $\overline{0.6} \in \mathcal{Q}_{\tau_2}(x_s, r)$ with $s > 0.4$ and $0 < r \leq \frac{2}{3}$, $\overline{0.6} = C_{\tau_1}(\overline{0.6}, r)$.

(2) Let $\tau_1 = \sigma_2$ and $\tau_2 = \sigma_3$. A fbts $(X, \tau_1, \tau_2)$ is not FP regular because for $\overline{0.3} \in \mathcal{Q}_{\tau_1}(x_t, r)$ with $t > 0.7$ and $0 < r \leq \frac{2}{3}$, for all $\lambda \in \mathcal{Q}_{\tau_1}(x_t, r) = \{\overline{1}, \overline{0.3}\}$ such that $\overline{0.3} < C_{\tau_2}(\lambda, r)$.

**Theorem 3.3.** *A fbts $(X, \tau_1, \tau_2)$ is FP regular iff $T_{\tau_j}^{\tau_i}(\lambda, r) = C_{\tau_i}(\lambda, r)$, for each $\lambda \in I^X$ and $r \in I_0$.*

**Proof.** From Theorem 1.7(5), we only show that $T_{\tau_j}^{\tau_i}(\lambda, r) \leq C_{\tau_i}(\lambda, r)$, for each $\lambda \in I^X$ and $r \in I_0$.

Suppose there exist $\lambda \in I^X$, $r \in I_0$ and $i, j \in \{1, 2\}$ with $i \neq j$ such that $T_{\tau_j}^{\tau_i}(\lambda, r) \not\leq C_{\tau_i}(\lambda, r)$. Then there exist $x \in X$ and $t \in I_0$ such that

$$T_{\tau_j}^{\tau_i}(\lambda, r)(x) > t > C_{\tau_i}(\lambda, r)(x). \tag{E}$$

Since $C_{\tau_i}(\lambda, r)(x) < t$, $x_t$ is not a r-$\tau_i$ cluster point of $\lambda$. Thus there exists $\mu \in \mathcal{Q}_{\tau_i}(x_t, r)$ such that

$$\lambda \leq \overline{1} - \mu.$$

Since $(X, \tau_1, \tau_2)$ is FP regular, for $\mu \in \mathcal{Q}_{\tau_i}(x_t, r)$, there exists $\rho \in \mathcal{Q}_{\tau_i}(x_t, r)$ such that

$$C_{\tau_j}(\rho, r) \leq \mu.$$

Thus

$$\lambda \leq \overline{1} - \mu \leq \overline{1} - C_{\tau_j}(\rho, r) = I_{\tau_j}(\overline{1} - \rho, r).$$

By Theorem 1.7(1)

$$T_{\tau_j}^{\tau_i}(\lambda, r) \leq \overline{1} - \rho.$$

Hence, $T_{\tau_j}^{\tau_i}(\lambda, r)(x) \leq (\overline{1} - \rho)(x) < t$. It is a contradiction for equation (E).

Conversely, for each $\mu \in \mathcal{Q}_{\tau_i}(x_t, r)$, $t > (\overline{1} - \mu)(x) = C_{\tau_i}(\overline{1} - \mu, r)(x)$. Since $T_{\tau_j}^{\tau_i}(\overline{1} - \mu, r) = C_{\tau_i}(\overline{1} - \mu, r)$, $x_t$ is not a r-$(\tau_i, \tau_j)$ θ-cluster point of $\overline{1} - \mu$. Then there exists $\rho \in \mathcal{Q}_{\tau_i}(x_t, r)$ such that $C_{\tau_j}(\rho, r) \leq \mu$. Thus, $(X, \tau_1, \tau_2)$ is FP regular.

**Theorem 3.4.** *Let $(X, \tau_1, \tau_2)$, $(Y, \eta_1, \eta_2)$ and $(Z, \gamma_1, \gamma_2)$ be fbts's. Let $f : X \to Y$ and $g : Y \to Z$ be functions. If $g$ and $f$ are FP strongly continuous (resp. FP continuous, FP θ-continuous), then $g \circ f$ is FP strongly continuous (resp. FP continuous, FP θ-continuous).*

**Proof.** Let $g$ and $f$ be FP strongly continuous. For each $\lambda \in I^X$ and $r \in I_0$,

$$\begin{aligned} g\left(f\left(T_{\tau_j}^{\tau_i}(\lambda, r)\right)\right) &\leq g(C_{\eta_i}(f(\lambda), r)) && (f \text{ is FP strongly continuous}) \\ &\leq g\left(T_{\eta_j}^{\eta_i}(f(\lambda), r)\right) && (\text{by Theorem 1.7(5)}) \\ &\leq C_{\gamma_i}(g(f(\lambda)), r). && (g \text{ is FP strongly continuous}) \end{aligned}$$

Thus, $g \circ f$ is FP strongly continuous. Others are similarly proved.

In general, the composition of two FP weakly θ-continuous functions need not be FP weakly θ-continuous from the following example.

**Example 3.5.** Define fuzzy topologies $\sigma_i : I^X \to I$ as in Example 2.7. Let $\tau_1 = \sigma_2$, $\tau_2 = \sigma_4$, $\eta_1 = \sigma_4$, $\eta_2 = \sigma_5$, $\gamma_1 = \sigma_3$, $\tau_2 = \sigma_4$. The identity functions $id_X : (X, \tau_1, \tau_2) \to (X, \eta_1, \eta_2)$ and $id_X : (X, \eta_1, \eta_2) \to (X, \gamma_1, \gamma_2)$ are FP weakly θ-continuous from Example 2.7(3), but

$id_X : (X, \tau_1, \tau_2) \to (X, \gamma_1, \gamma_2)$ is not FP weakly θ-continuous because

$$\overline{0.7} = C_{\tau_1}\left(\overline{0.5}, \frac{2}{3}\right) > T^{\gamma_1}_{\gamma_2}\left(\overline{0.5}, \frac{2}{3}\right) = \overline{0.6}.$$

From Theorem 3.3, we can obtain the following theorem.

**Theorem 3.6.** *Let* $(X, \tau_1, \tau_2)$ *and* $(Y, \eta_1, \eta_2)$ *be fbts's. Let* $f : (X, \tau_1, \tau_2) \to (Y, \eta_1, \eta_2)$ *be a function.*

(1) *If* $(X, \tau_1, \tau_2)$ *is FP regular, then*

(a) *f is FP strongly continuous iff f is FP continuous.*

(b) *f is FP* θ*-continuous iff f is FP weakly* θ*-continuous.*

(2) *If* $(Y, \eta_1, \eta_2)$ *is FP regular, then*

(a) *f is FP strongly continuous iff f is FP* θ*-continuous.*

(b) *f is FP continuous iff f is FP weakly* θ*-continuous.*

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# STRONGLY GENERALIZED CLOSED SETS IN TOPOLOGICAL SPACES

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## Abstract

In this paper, we introduce the concepts of strongly generalized closed sets and strongly generalized open sets, which are generalizations of closed sets and open sets in topological spaces. Further, we introduce $T_s$ and $T_p$ spaces. Also, we introduce a new closure operator $C^s$ and we obtain a new topology $\tau^s$ and study some of their properties.

## 1. Introduction

In this paper, we introduce a new generalization of closed sets namely 'strongly generalized closed sets', which is stronger than the generalized closed sets introduced by Levine [11] in 1970. Further, we introduce $T_p$ and $T_s$ spaces and study some of their properties.

Throughout this paper $X$, $Y$ and $Z$ are topological spaces on which no separation axioms are assumed unless otherwise explicitly stated. For a subset $A$ of a topological space $X$, $\text{int}(A)$, $cl(A)$, $scl(A)$ and $A^c$ denote the interior, closure, semi-closure and the complement of $A$, respectively.

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## 2. Preliminaries

Here we recall the following known definitions.

**Definition 2.1.** A subset $A$ of a topological space $X$ is called:

(a) *generalized closed* (*g-closed*) [11] if $cl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is open in $X$.

(b) *semi generalized closed* (*sg-closed*) [4] if $scl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is semi-open in $X$.

(c) *generalized semi-closed* (*gs-closed*) [3] if $scl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is open in $X$.

(d) *semi-closed* [5] if $\text{int}(cl(A)) \subseteq A$.

(e) $\alpha$-*closed* [15] if $cl(\text{int}(cl(A))) \subseteq A$.

(f) *pre closed* [14] if $cl(\text{int}(A)) \subseteq A$.

(g) $\beta$-*closed* [1] if $\text{int}(cl(\text{int}(A))) \subseteq A$.

(h) *regular generalized closed* (*rg-closed*) [16] if $cl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is regular open in $X$.

(i) *generalized semi-preclosed* (*gsp-closed*) [7] if $spcl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is open in $X$.

(j) *generalized* $\alpha$-*closed* ($g\alpha$-*closed*) [13] set if $cl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is $\alpha$-open in $X$.

(k) $\alpha$-*generalized closed* ($\alpha g$-*closed*) [12] set if $\alpha cl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is open in $X$.

(l) *weakly generalized closed* (*wg-closed*) [18] if $cl(\text{int}(A)) \subseteq G$ whenever $A \subseteq G$ and $G$ is open in $X$.

The complements of the above-mentioned sets are called their respective *open* sets.

**Definition 2.2.** A topological space $X$ is called:

(a) $T_{1/2}$*-space* [11] if every $g$-closed set in $X$ is closed in $X$.

(b) $T_b$ *space* [6] if every $gs$-closed set in $X$ is closed in $X$.

(c) $T_d$ *space* [6] if every $gs$-closed set in $X$ is $g$-closed in $X$.

**Definition 2.3.** A map $f : X \to Y$ is called *gc-irresolute* [17] if $f^{-1}(V)$ is $g$-closed in $X$ for every $g$-closed set $V$ in $Y$.

**Definition 2.4.** $C^*(E) =$ the intersection of all $g$-closed sets containing $E$ in $X$. $\tau^* = \{F : C^*(F^c) = F^c\}$ [9].

## 3. Strongly Generalized Closed Sets in Topological Spaces

In this section, we introduce the concept of strongly generalized closed sets in topological spaces.

**Definition 3.1.** A subset $A$ of a topological space $X$ is called *strongly generalized closed set* (*strongly g-closed*) if $cl(A) \subseteq G$, whenever $A \subseteq G$ and $G$ is $g$-open in $X$.

**Theorem 3.2.** *Every closed set in $X$ is strongly g-closed in $X$ but not conversely.*

**Proof.** Straightforward.

The converse of the above theorem need not be true as is seen from the following example.

**Example 3.3.** Consider the topological space $X = \{a, b, c\}$ with topology $\tau = \{\varnothing, X, \{a\}, \{a, b\}\}$. The set $\{a, c\}$ is strongly $g$-closed but not closed.

**Theorem 3.4.** *Union of two strongly g-closed sets in $X$ is strongly g-closed in $X$.*

**Proof.** Straightforward.

**Theorem 3.5.** *A subset $A$ of $X$ is strongly g-closed if and only if $cl(A) - A$ contains no nonempty g-closed set in $X$.*

**Proof.** Suppose that $F$ is a nonempty $g$-closed subset of $cl(A) - A$. Now $F \subseteq cl(A) - A$. Then $F \subseteq cl(A) \cap A^c$, since $cl(A) - A = cl(A) \cap A^c$. Therefore $F \subseteq cl(A)$ and $F \subseteq A^c$. Since $F^c$ is $g$-open set and $A$ is strongly $g$-closed, $cl(A) \subseteq F^c$. That is $F \subseteq (cl(A))^c$. Hence $F \subseteq cl(A) \cap [cl(A)]^c = \emptyset$. That is $F = \emptyset$. Thus $cl(A) - A$ contains no nonempty $g$-closed set.

Conversely, assume that $cl(A) - A$ contains no nonempty $g$-closed set. Let $A \subseteq G$, $G$ is $g$-open. Suppose that $cl(A)$ is not contained in $G$. Then $cl(A) \cap G^c$ is a nonempty $g$-closed set [11] of $cl(A) - A$, which is a contradiction. Therefore $cl(A) \subseteq G$ and hence $A$ is strongly $g$-closed.

**Theorem 3.6.** *If a subset $A$ of $X$ is strongly g-closed and $A \subseteq B \subseteq cl(A)$, then $B$ is strongly g-closed in $X$.*

**Proof.** $cl(B) - B \subseteq cl(A) - A$ and since $cl(A) - A$ contains no nonempty $g$-closed set, neither does $cl(B) - B$. By Theorem 3.5, the result follows.

**Theorem 3.7.** *Every strongly g-closed set in $X$ is a g-closed set in $X$ but not conversely.*

**Proof.** Let $A$ be a strongly $g$-closed set. Let $A \subseteq G$, where $G$ is open. Since every open set is $g$-open and $A$ is strongly $g$-closed, we have $cl(A) \subseteq G$. Therefore $A$ is $g$-closed.

The converse need not be true as is seen from the following example.

**Example 3.8.** Consider the topological space $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{a\}\}$. The set $\{b\}$ is $g$-closed but not strongly $g$-closed.

**Remark 3.9.** If $A$ is strongly $g$-closed in $X$, then it is not only $g$-closed in $X$ but also $\alpha g$-closed, $gs$-closed, $gsp$-closed, $rg$-closed and $wg$-closed.

**Remark 3.10.** The concept of strongly $g$-closed set is independent of the following class of sets namely semi-closed set, preclosed set, $\alpha$-closed set, $\beta$-closed set, $g\alpha$-closed set and $sg$-closed set.

**Example 3.11.** Consider the topological space $X = \{a, b, c\}$ with topology $\tau = \{\varnothing, X, \{a\}, \{a, b\}\}$. In this space, the set $\{b\}$ is semiclosed, preclosed, $\alpha$-closed, $\beta$-closed, $g\alpha$-closed and $sg$-closed but not strongly $g$-closed. Also the set $\{a, c\}$ is strongly $g$-closed but not any of the sets mentioned above.

**Definition 3.12.** A subset $A$ of a topological space $X$ is called *strongly generalized open set* (*strongly g-open*) if $A^c$ is strongly $g$-closed.

**Theorem 3.13.** *A subset $A$ of $X$ is strongly g-open in $X$ if and only if $F \subseteq \text{int}(A)$ whenever $F \subseteq A$ and $F$ is g-closed in $X$.*

**Proof.** Straightforward.

**Theorem 3.14.** *For each $x \in X$, $\{x\}$ is g-closed in $X$ or $\{x\}^c$ is strongly g-closed in $X$.*

**Proof.** If $\{x\}$ is not $g$-closed, then the only $g$-open set containing $\{x\}^c$ is $X$. Also the closure of $\{x\}^c$ is contained in $X$ and hence $\{x\}^c$ is strongly $g$-closed in $X$.

## 4. $T_p$ and $T_s$ Spaces

In this section, we introduce two new classes of topological spaces called *$T_p$ space* and *$T_s$ space.*

**Definition 4.1.** A topological space $X$ is called *$T_p$ space* if every strongly $g$-closed set is closed in $X$.

**Definition 4.2.** A topological space $X$ is called *$T_s$ space* if every $g$-closed set is strongly $g$-closed in $X$.

**Theorem 4.3.** *If $X$ is $T_{1/2}$, then it is $T_p$ but not conversely.*

**Proof.** Let $X$ be a $T_{1/2}$-space. Since every strongly $g$-closed set is $g$-closed and $X$ is $T_{1/2}$, $X$ is $T_p$.

The converse need not be true as is seen from the following example.

**Example 4.4.** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. Then $(X, \tau)$ is $T_p$ space but not $T_{1/2}$.

**Remark 4.5.** The spaces $T_0$ and $T_p$ are independent as is seen from the following example.

**Example 4.6.** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b, c\}\}$. Then the space $(X, \tau)$ is a $T_0$ space but not a $T_p$ space and the space $(X, \sigma)$ is a $T_p$ space but not a $T_0$ space.

**Theorem 4.7.** *If X is $T_{1/2}$, then it is $T_s$ but not conversely.*

**Proof.** Let $X$ be a $T_{1/2}$ space. Let $A$ be any $g$-closed set in $T_s$. Since $X$ is $T_{1/2}$, $A$ is closed in $X$. Since every closed set is strongly $g$-closed, $A$ is strongly $g$-closed. Hence $X$ is $T_s$.

The converse need not be true as is seen from the following example.

**Example 4.8.** Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, X, \{a, b\}\}$. Then $X$ is $T_s$ but not $T_{1/2}$.

**Remark 4.9.** $T_s$ and $T_p$ spaces are independent as is seen from the following example.

**Example 4.10.** Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{a, b\}\}$. Then $(X, \tau_1)$ is $T_p$ but not $T_s$. Also $(X, \tau_2)$ is $T_s$ but not $T_p$.

**Remark 4.11.** A $T_s$ space need not be $T_0$. For, consider the topological space $X = \{a, b, c\}$ with indiscrete topology. Then $X$ is $T_s$ but not $T_0$.

**Theorem 4.12.** *If X is $T_b$ space, then it is $T_p$ but not conversely.*

**Proof.** Let $X$ be a $T_b$ space. Let $A$ be any strongly $g$-closed set in $X$. Then $A$ is $gs$-closed in $X$. Since $X$ is $T_b$, $A$ is closed in $X$. Hence $X$ is a $T_p$ space.

The converse need not be true as is seen from the following example.

**Example 4.13.** Let $X = \{a, b, c\}$ with $\tau = \{\varnothing, X, \{a\}\}$. Then $X$ is $T_p$ but not $T_b$.

**Remark 4.14.** $T_s$ and $T_d$ spaces are independent as is seen from the following example.

**Example 4.15.** Let $X = \{a, b, c\}$, $\tau_1 = \{\varnothing, X, \{a\}\}$ and $\tau_2 = \{\varnothing, X, \{a\}, \{a, b\}\}$. Then $(X, \tau_1)$ is $T_d$ but not $T_s$. Also $(X, \tau_2)$ is $T_s$ but not $T_d$.

**Theorem 4.16.** *If $X$ is a $T_b$ space, then it is $T_s$ but not conversely.*

**Proof.** Let $X$ be a $T_b$ space. Let $A$ be any $g$-closed set in $X$. Then $A$ is $gs$-closed in $X$. Since $X$ is $T_b$, $A$ is closed in $X$ and hence strongly $g$-closed in $X$. Therefore $X$ is $T_s$.

The converse of the above theorem need not be true as is seen from the following example.

**Example 4.17.** Let $X = \{a, b, c\}$ with $\tau = \{\varnothing, X, \{a, b\}\}$. Then $X$ is $T_s$ but not $T_b$.

**Remark 4.18.** $T_b$ and $T_p$ spaces are independent as is seen from the following example.

**Example 4.19.** Let $X = \{a, b, c\}$, $\tau_1 = \{\varnothing, X, \{a, b\}\}$ and $\tau_2 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. Then $(X, \tau_1)$ is $T_d$ but not $T_p$. Also $(X, \tau_2)$ is $T_p$ but not $T_d$.

**Theorem 4.20.** *$X$ is $T_{1/2}$ if and only if $X$ is both $T_p$ and $T_s$.*

**Proof.** From Theorems 4.3 and 4.7, necessity follows. For the sufficiency, let $A$ be any $g$-closed set in $X$. Since $X$ is $T_s$, $A$ is strongly $g$-closed. Since $X$ is $T_p$, $A$ is closed in $X$. Therefore $X$ is $T_{1/2}$.

**Theorem 4.21.** *If $A$ is strongly $g$-closed set in $X$ and if $f : X \to Y$ is gc-irresolute and closed, then $f(A)$ is strongly $g$-closed in $Y$.*

**Proof.** If $f(A) \subseteq G$ where $G$ is $g$-open in $Y$, then $A \subseteq f^{-1}(G)$ and hence $cl(A) \subseteq f^{-1}(G)$. Thus $f(cl(A)) \subseteq G$ and $f(cl(A))$ is a closed set. It follows that $cl(f(A)) \subseteq cl(f(cl(A))) = f(cl(A)) \subseteq G$. Therefore $f(A)$ is strongly $g$-closed.

**Theorem 4.22.** *$X$ is $T_p$ if and only if for each $x \in X$, $\{x\}$ is $g$-closed or open.*

**Proof.** Suppose that $X$ is $T_p$ and for each $x \in X$, $\{x\}$ is not $g$-closed. Since $X$ is the only $g$-open set containing $\{x\}^c$, $\{x\}^c$ is strongly $g$-closed and thus closed. Hence $\{x\}$ is open.

To prove the converse assume that $A$ is strongly $g$-closed set in $X$. Let $x \in cl(A)$.

**Case (i)** If $\{x\}$ is $g$-closed and if $x \notin A$, then $\{x\} \subseteq cl(A) - A$. This contradicts Theorem 3.5. Therefore $x \in A$.

**Case (ii)** If $\{x\}$ is open, then $\{x\} \cap A \neq \emptyset$ and so $x \in A$.

In either case, $x \in A$ and hence $A$ is closed.

**Theorem 4.23.** *If $f : X \to Y$ is a bijection, continuous and closed and if $B$ is a strongly $g$-closed (strongly $g$-open) subset of $Y$, then $f^{-1}(B)$ is strongly $g$-closed (strongly $g$-open) in $X$.*

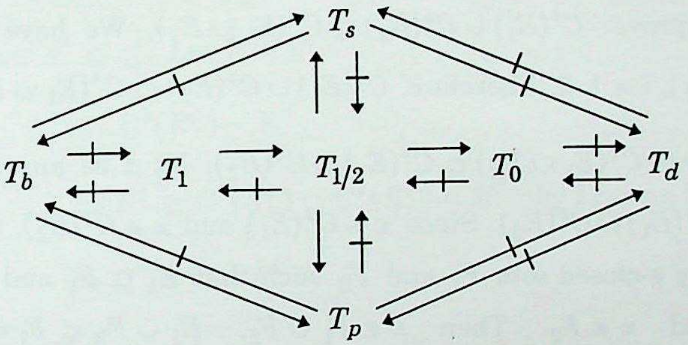
**Proof.** Suppose that $B$ is a strongly $g$-closed subset of $Y$ and that $f^{-1}(B) \subseteq G$ where $G$ is $g$-open in $X$. We will show that $cl(f^{-1}(B)) \subseteq G$ or that $cl(f^{-1}(B)) \cap G^c = \emptyset$. Now $f(cl(f^{-1}(B)) \cap G^c) \subseteq cl(B) - B$ and by Theorem 3.5 $f(cl(f^{-1}(B))) \cap G^c = \emptyset$. Thus $cl(f^{-1}(B)) \cap G^c = \emptyset$. By taking complements, we can show that if $B$ is strongly $g$-open in $Y$, then $f^{-1}(B)$ is strongly $g$-open in $X$.

We can show that if $B$ is strongly $g$-open in $Y$, then $f^{-1}(B)$ is strongly $g$-open in $X$.

**Theorem 4.24.** *If* $X$ *is* $T_p$ *with* $Y \subseteq X$, *then* $Y$ *is* $T_p$.

**Proof.** For $y \in Y$, $\{y\}$ is open or $g$-closed in $X$. Using Theorem 4.22 and [11, Theorem 2.9], $\{y\}$ is open or $g$-closed in $Y$.

**Remark 4.25.** In [11], Levine proved that $T_{1/2}$ lies strictly between $T_0$ and $T_1$ spaces. We obtain the following diagram:




## 5. Strongly Generalized Closure Operator

In this section, we introduce a new closure operator and a new topology using this closure operator and study some of their properties.

**Definition 5.1.** For a space $(X, \tau)$, let $D^s = \{A : A \subseteq X$ and $A$ is strongly $g$-closed$\}$.

**Definition 5.2.** For any $E \subseteq X$, define $C^s(E) = \cap\{A : E \subseteq A \in D^s\}$.

**Lemma 5.3.** *If* $E \subseteq X$, *then* $E \subseteq C^*(E) \subseteq C^s(E) \subseteq cl(E)$.

**Proof.** Since every closed set is strongly $g$-closed and since every strongly $g$-closed set is $g$-closed, the proof follows.

**Remark 5.4.** The containment relations in the Lemma 5.3 may be proper. Consider the topological space $X = \{a, b, c\}$ with topology $\tau = \{\varnothing, X, \{a, b\}\}$. Since $C^s(\{a\}) = \{a, c\}$ and $cl(\{a\}) = X$, we have

$\{a\} \subset C^s(\{a\}) \subset cl(\{a\})$. Also, in this space $X = \{a, b, c\}$ with topology $\tau = \{\varnothing, X, \{a\}\}$, we have $C^*(\{c\}) = \{c\}$ and $C^s(\{c\}) = \{b, c\}$. Therefore $\{c\} \subset C^*(\{c\}) \subset C^s(\{c\})$.

**Theorem 5.5.** *$C^s$ is a Kuratowski closure operator on $X$.*

**Proof.** (a) $C^s(\varnothing) = \varnothing$ and $E \subseteq C^s(E)$ follows from Lemma 5.3.

(b) To prove: $C^s(E_1) \cup C^s(E_2) \subseteq C^s(E_1 \cup E_2)$. We have $C^s(E_i) \subseteq C^s(E_1 \cup E_2)$, $i = 1, 2$. Therefore, $C^s(E_1) \cup C^s(E_2) \subseteq C^s(E_1 \cup E_2)$.

To prove: $C^s(E_1 \cup E_2) \subseteq C^s(E_1) \cup C^s(E_2)$, let $x$ be any point such that $x \notin C^s(E_1) \cup C^s(E_2)$. Since $x \notin C^s(E_1)$ and $x \notin C^s(E_2)$, there exist two strongly $g$-closed sets $F_1$ and $F_2$ such that $E_1 \subseteq F_1$ and $E_2 \subseteq F_2$, $x \notin F_1$ and $x \notin F_2$. Then $x \notin F_1 \cup F_2$, $E_1 \cup E_2 \subseteq F_1 \cup F_2$ and $F_1 \cup F_2$ is strongly $g$-closed by Theorem 3.4. Thus we have $x \notin C^s(E_1 \cup E_2)$. Therefore, we have $C^s(E_1 \cup E_2) \subseteq C^s(E_1) \cup C^s(E_2)$. Hence $C^s(E_1 \cup E_2) = C^s(E_1) \cup C^s(E_2)$.

(c) Let $A$ be any strongly $g$-closed set in $X$ containing $E$. Then by definition $C^s(E) \subseteq A$. Since $A$ is strongly $g$-closed and contains $C^s(E)$, $C^s(C^s(E)) \subseteq A$. This means that $C^s(C^s(E))$ is contained in every strongly $g$-closed set containing $E$. Hence $C^s(C^s(E)) \subseteq C^s(E)$. Therefore, $C^s(C^s(E)) = C^s(E)$.

**Definition 5.6.** Let $\tau^s$ be the topology in $X$ generated by $C^s$ in the usual manner. That is $\tau^s = \{F : C^s(F^c) = F^c\}$.

**Theorem 5.7.** *For any space $(X, \tau)$, the following hold:*

(i) $\tau \subseteq \tau^s \subseteq \tau^*$.

(ii) *The space $(X, \tau)$ is $T_p$ if and only if $\tau = \tau^s$.*

**Proof.** To prove: $\tau \subseteq \tau^s$. Let $E \in \tau$.

$$E^c \text{ is closed in } \tau$$

$$\Rightarrow E^c \subseteq C^s(E^c) \subseteq cl(E^c) = E^c \quad \text{by Lemma 5.3}$$

$$\Rightarrow C^s(E^c) \subseteq E^c.$$

But $E^c \subseteq C^s(E^c)$ implies $C^s(E^c) = E^c$ and hence $E \in \tau^s$.

To prove: $\tau^* \subseteq \tau^s$. Let $E \in \tau^s$.

$$C^s(E^c) = E^c$$

$$\Rightarrow E^c \subseteq C^*(E^c) \subseteq C^s(E^c) = E^c \quad \text{by Lemma 5.3}$$

$$\Rightarrow C^*(E^c) = E^c$$

$$\Rightarrow E \in \tau^*.$$

Therefore $\tau \subseteq \tau^s \subseteq \tau^*$.

(ii) Assume that $\tau = \tau^s$ and let $A \subseteq X$ be strongly $g$-closed in $(X, \tau)$. Then $A = C^s(A)$ and so $A^c \in \tau^s \Rightarrow A^c \in \tau = \tau$. Thus $(X, \tau)$ is $T_p$. Conversely, assume that $(X, \tau)$ is $T_p$. Then every strongly $g$-closed set is closed in $X$. Therefore $\tau^s \subseteq \tau$ and hence $\tau = \tau^s$.

**Theorem 5.8.** *If a space $(X, \tau)$ is $T_{1/2}$, then $(X, \tau^s)$ is $T_{1/2}$.*

**Proof.** Let $x \in X$ be any point. By Theorem 3.14, $\{x\}$ is $g$-closed in $(X, \tau)$ or $\{x\}^c$ is strongly $g$-closed in $(X, \tau)$. If $\{x\}$ is $g$-closed, then $\{x\}$ is $\tau$-closed and hence $\{x\}$ is $\tau^s$-closed. If $\{x\}^c$ is strongly $g$-closed in $(X, \tau)$, by the definition of $\tau^s$, $\{x\}$ is $\tau^s$-open. Therefore $(X, \tau^s)$ is $T_{1/2}$ by Theorem 2.5 of [8].

**Corollary 5.9.** *For a $T_{1/2}$-space $(X, \tau)$, $(\tau^s)^s = \tau^s$.*

**Proof.** By Theorem 5.8, $(X, \tau^s)$ is $T_{1/2}$. But every $T_{1/2}$ space is $T_p$. Thus, $(\tau^s)^s = \tau^s$ by Theorem 5.7.

We characterize the discreteness of $(X, \tau^s)$.

**Theorem 5.10.** *The following conditions are equivalent*:

(a) $(X, \tau^s)$ *is discrete.*

(b) *For each* $x \in X$, $\{x\}^c$ *is strongly g-closed in* $(X, \tau)$.

(c) *If* $\{x\}$ *is g-closed in* $(X, \tau)$, $\{x\}$ *is open in* $(X, \tau)$.

**Proof.** (a) $\Rightarrow$ (b) If $(X, \tau^s)$ is discrete, then, for each $x$, $\{x\}^c = C^s(\{x\}^c) = \cap\{A : \{x\}^c \subseteq A \in D^s\}$ it follows that $\{x\}^c$ is itself strongly $g$-closed in $(X, \tau)$.

(b) $\Rightarrow$ (c) Suppose $\{x\}$ is $g$-closed in $(X, \tau)$. Then $\{x\}^c$ is $g$-open and strongly $g$-closed in $(X, \tau)$ by (b). Thus we have $cl(\{x\}^c) \subseteq \{x\}^c$. Therefore, $\{x\}$ is open in $(X, \tau)$.

(c) $\Rightarrow$ (a) Let $x \in X$ be any point. If $\{x\}$ is $g$-closed in $(X, \tau)$, then $\{x\} \in \tau$ and thus $\{x\} \in \tau^s$ by Theorem 5.7. If $\{x\}$ is not $g$-closed in $(X, \tau)$, $\{x\}^c$ is strongly $g$-closed in $(X, \tau)$. Again we have $\{x\} \in \tau^s$.

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# STRONGLY GENERALIZED SEMI-PRECLOSED SETS AND STRONGLY SEMIPRE-$T_{1/2}$ SPACE

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## Abstract

In this paper, we introduce the concepts of strongly generalized semi-preclosed sets, strongly generalized semi-preopen sets and strongly semipre-$T_{1/2}$ spaces in topological spaces, which are stronger forms of generalized semi-preclosed sets, generalized semi-preopen sets and semipre-$T_{1/2}$ spaces, respectively. Further, we study some of their properties.

## 1. Introduction

Levine [10] introduced the concept of generalized closed ($g$-closed) sets in topological spaces and a class of topological spaces called $T_{1/2}$-*spaces*. Dontchev [7] introduced the concepts of generalized semi-preclosed ($gsp$-closed) sets and semipre-$T_{1/2}$ spaces.

In this paper, we introduce and study the concepts of two new classes of sets, namely strongly generalized semi-preclosed (strongly $gsp$-closed) sets and strongly generalized semi-preopen (strongly $gsp$-open) sets which are stronger forms of generalized semi-preclosed ($gsp$-closed) sets

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and generalized semi-preopen (*gsp*-open) sets. Also, we introduce the concept of strongly semipre-$T_{1/2}$ spaces. Further, we study some of their properties.

Throughout this paper $X$, $Y$ and $Z$ denote topological spaces on which no separation axioms are assumed unless otherwise explicitly stated. For a subset $A$ of a topological space $X$, $\operatorname{int}(A)$, $cl(A)$, $scl(A)$, $spcl(A)$, $sp\operatorname{int}(A)$ and $A^c$ denote the interior, closure, semi-closure, semi-preclosure, semi-preinterior and the complement of $A$, respectively.

## 2. Preliminaries

Here we recall the following known definitions and results.

**Definition 2.1.** A subset $A$ of a topological space $X$ is called:

(a) *generalized closed* (*g-closed*) [10] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

(b) *generalized semi-closed* (*gs-closed*) [3] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

(c) *pre closed* [13] if $cl(\operatorname{int}(A)) \subseteq A$.

(d) *semi-closed* [5] if $\operatorname{int}(cl(A)) \subseteq A$.

(e) *α-closed* [14] if $cl(\operatorname{int}(cl(A))) \subseteq A$.

(f) *semi-preclosed* [2] if $\operatorname{int}(cl(\operatorname{int}(A))) \subseteq A$.

(g) *regular closed* [16] if $cl(\operatorname{int}(A)) = A$.

(h) *α-generalized closed* (*αg-closed*) [11] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

(i) *regular generalized closed* (*rg-closed*) [15] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $X$.

(j) *generalized semi-preclosed* (*gsp-closed*) [7] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

(k) *weakly generalized closed* (*wg-closed*) [17] if $cl(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

(l) *semi-generalized closed* (*sg-closed*) [4] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semiopen in $X$.

(m) *Q-set* [8] if $\text{int}(cl(A)) = cl(\text{int}(A))$.

The complements of the above-mentioned sets are called their respective *open* sets.

**Definition 2.2.** A topological space $X$ is called:

(a) a $T_{1/2}$*-space* [10] if every $g$-closed set in $X$ is closed in $X$.

(b) a *semipre-*$T_{1/2}$ [7] if every $gsp$-closed set in $X$ is semi-preclosed in $X$.

(c) an $\alpha$*-space* [14] if every $\alpha$-closed set in $X$ is closed in $X$.

(d) a *semi-*$T_{1/2}$ [4] if every $sg$-closed set in $X$ is semiclosed in $X$.

(e) a $T_d$*-space* [6] if every $gs$-closed set in $X$ is $g$-closed in $X$.

**Lemma 2.3.** *For a subset $A$ of $X$,*

(a) $spcl(A) = A \cup \text{int}(cl(\text{int}(A)))$ [2].

(b) $spcl(A^c) = (sp\,\text{int}(A))^c$ [2].

## 3. Basic Properties of Strongly Generalized Semi-preclosed Sets

In this section, we introduce strongly generalized semi-preclosed sets and study some of their properties.

**Definition 3.1.** A subset $A$ of $X$ is called *strongly generalized semi-preclosed set* (*strongly gsp-closed set*) if $spcl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is $g$-open in $X$.

**Theorem 3.2.** *Every semi-preclosed set in $X$ is strongly gsp-closed in $X$.*

**Proof.** Let $A$ be a semi-preclosed set in $X$. Let $U$ be a $g$-open set in $X$ containing $A$. A set $A \subseteq X$ is semi-preclosed if and only if $spcl(A) = A$ [7, Theorem 3.1]. Therefore, $spcl(A) \subseteq U$. Thus $A$ is strongly $gsp$-closed in $X$.

The converse of the above theorem need not be true as is seen from the following example.

**Example 3.3.** Consider the topological space $X = \{a, b, c\}$ with topology $\tau = \{X, \varnothing, \{a\}, \{a, c\}\}$. The set $\{a, b\}$ is strongly $gsp$-closed but not semi-preclosed.

**Remark 3.4.** Union of two strongly $gsp$-closed sets need not be strongly $gsp$-closed. Consider the topological space $X = \{a, b, c, d\}$ with topology $\tau = \{X, \varnothing, \{a, b\}\}$. The sets $\{a\}$ and $\{b, c\}$ are strongly $gsp$-closed sets but their union $\{a, b, c\}$ is not strongly $gsp$-closed.

**Theorem 3.5.** *Every strongly gsp-closed set in X is gsp-closed in X.*

**Proof.** Let $A$ be strongly $gsp$-closed set in $X$. Let $G$ be an open set in $X$ containing $A$. Since every open set is $g$-open and $A$ is strongly $gsp$-closed, we have $spcl(A) \subseteq G$. Therefore $A$ is $gsp$-closed.

The converse of the above theorem need not be true as is seen from the following example.

**Example 3.6.** Consider the topological space $X = \{a, b, c\}$ with topology $\tau = \{X, \varnothing, \{a\}\}$. The set $\{a, b\}$ is $gsp$-closed but not strongly $gsp$-closed.

From the above results we have the following diagram

$$\text{semi-preclosed set} \;\underset{\not\longleftarrow}{\longrightarrow}\; \text{strongly } gsp\text{-closed set} \;\underset{\not\longleftarrow}{\longrightarrow}\; gsp\text{-closed set}$$

**Remark 3.7.** The concept of strongly $gsp$-closed set is independent of the following classes of sets namely $g$-closed, $\alpha g$-closed, $gs$-closed, $rg$-closed, $wg$-closed as is seen from Examples 3.8 and 3.9.

**Example 3.8.** Consider the topological space $X = \{a, b, c, d\}$ with topology $\tau = \{X, \emptyset, \{a, b\}\}$. In this topology the set $\{a, b, c\}$ is $g$-closed, $\alpha g$-closed and $gs$-closed but not strongly $gsp$-closed. The set $\{a\}$ is strongly $gsp$-closed but not $g$-closed, $\alpha g$-closed or $gs$-closed.

**Example 3.9.** Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\sigma = \{X, \emptyset, \{a\}\}$. In $(X, \tau)$, the set $\{a\}$ is strongly $gsp$-closed but neither $wg$-closed nor $rg$-closed. Also, the set $\{a, b\}$ in $(X, \tau)$ is $rg$-closed but not strongly $gsp$-closed. In $(X, \sigma)$, the set $\{a, b\}$ is $wg$-closed but not strongly $gsp$-closed.

**Theorem 3.10.** *Let $A$ be a strongly gsp-closed set in $X$. Then $spcl(A) - A$ contains no nonempty g-closed set.*

**Proof.** Suppose that $F$ is a $g$-closed subset of $spcl(A) - A$. This implies $F \subseteq spcl(A)$ and $F \subseteq A^c$. Since $F^c$ is $g$-open set and $A$ is strongly $gsp$-closed, $spcl(A) \subseteq F^c$. Therefore $F \subseteq spcl(A) \cap (spcl(A))^c$ $= \emptyset$. Hence $spcl(A) - A$ contains no nonempty $g$-closed set.

**Theorem 3.11.** *If $A$ is g-open and strongly gsp-closed in $X$, then $A$ is semi-preclosed.*

**Proof.** Since $A$ is $g$-open and strongly $gsp$-closed, $spcl(A) \subseteq A$. But always $A \subseteq spcl(A)$. Therefore $A = spcl(A)$. Thus $A$ is semi-preclosed.

**Corollary 3.12.** *If $A$ is open and strongly gsp-closed set in $X$, then $A$ is semi-preclosed.*

**Proof.** Since every open set is $g$-open, the result follows.

**Lemma 3.13.** *Let $G \subseteq A \subseteq X$. If $G$ is g-open in $X$ and $A$ is clopen in $X$, then $G$ is g-open in $A$.*

**Proof.** Let $F$ be a closed subset of $A$ such that $F \subseteq G$. Since $A$ is closed in $X$, $F$ is closed in $X$. Since $G$ is $g$-open in $X$, $F \subseteq \text{int}(G)$. Since $A$ is open in $X$, $\text{int}_A(G) = \text{int}(G) \cap A$ holds. Therefore, we have $F = F \cap A \subseteq \text{int}_A(G)$ and so $G$ is $g$-open in $A$.

**Theorem 3.14.** *Let $F \subseteq A \subseteq X$. If F is strongly gsp-closed in A and A is clopen, then F is strongly gsp-closed in X.*

**Proof.** Let $U$ be $g$-open in $X$ and $F \subseteq U$. Then $F \subseteq U \cap A$ and $U \cap A$ is $g$-open in $X$. By Lemma 3.13, $U \cap A$ is $g$-open in $A$. Since $F$ is strongly $gsp$-closed in $A$, $spcl_A(F) \subseteq U \cap A$. Since $A$ is closed in $X$, $spcl(A) = A$. Using [7, Lemma 3.16], we have $spcl(F) = spcl(A \cap F) \subseteq spcl(A) \cap spcl(F) = A \cap spcl(F) = spcl_A(F) \subseteq U \cap A$ and so $spcl(F) \subseteq U$. This implies that $F$ is strongly $gsp$-closed in $X$.

**Theorem 3.15.** *Let $F \subseteq A \subseteq X$. If F is strongly gsp-closed in X and A is open, then F is strongly gsp-closed in A.*

**Proof.** Let $V$ be a $g$-open subset in $A$ such that $F \subseteq V$. Since $A$ is $g$-open in $X$ and $V$ is $g$-open in $A$, $V$ is $g$-open in $X$ by [10, Theorem 4.6]. Since $F$ is strongly $gsp$-closed in $X$, then $spcl(F) \subseteq V$. Thus, by [7, Lemma 3.16], $spcl_A(F) = spcl(F) \cap A \subseteq V \cap A = V$. This shows that $F$ is strongly $gsp$-closed in $A$.

**Corollary 3.16.** *Let $F \subseteq A \subseteq X$, where A is clopen in X. Then F is strongly gsp-closed in A if and only if F is strongly gsp-closed in X.*

**Proof.** First assume that $F$ is strongly $gsp$-closed in $A$. By Theorem 3.14, $F$ is strongly $gsp$-closed in $X$. Conversely, assume that $F$ is strongly $gsp$-closed in $X$. By Theorem 3.15, $F$ is strongly $gsp$-closed in $A$.

**Theorem 3.17.** *For a subset $A \subseteq X$ the following conditions are equivalent:*

(1) *A is open and strongly gsp-closed.*

(2) *A is regular open.*

**Proof.** (1) $\Rightarrow$ (2) Since $A$ is open and strongly $gsp$-closed, $spcl(A) \subseteq A$ and so $A \cup \text{int}(cl(\text{int}(A))) \subseteq A$. Since $A$ is open, we have $\text{int}(cl(A)) \subseteq A$. Since every open set is preopen, $A \subseteq \text{int}(cl(A))$. Therefore $A = \text{int}(cl(A))$. Thus $A$ is regular open.

(2) $\Rightarrow$ (1) Every regular open set is open. Let $U$ be a $g$-open set in $X$

containing $A$. Now $\text{int}(cl(\text{int}(A))) \subseteq \text{int}(cl(A)) = A$, since $A$ is regular open. Therefore $A \cup \text{int}(cl(\text{int}(A))) \subseteq A \subseteq U$. That is $spcl(A) \subseteq U$. Hence $A$ is strongly $gsp$-closed.

**Theorem 3.18.** *For a subset $A \subseteq X$ the following conditions are equivalent*:

(1) *$A$ is clopen,*

(2) *$A$ is open, a $Q$-set and strongly $gsp$-closed.*

**Proof.** (1) $\Rightarrow$ (2) Since $A$ is clopen, $A$ is both open and a $Q$-set. Let $G$ be a $g$-open set in $X$ containing $A$. Now, $\text{int}(cl(\text{int}(A))) = A \subseteq G$. Therefore $spcl(A) \subseteq G$. Hence $A$ is strongly $gsp$-closed in $X$.

(2) $\Rightarrow$ (1) By Theorem 3.17, $A$ is regular open. Since a regular open set is open, $A$ is open. Since $A$ is regular open and a $Q$-set, $A$ is closed. Hence $A$ is closed and open.

## 4. Strongly Generalized Semi-preopen Sets

In this section, we introduce strongly generalized semi-preopen sets and study some of their properties.

**Definition 4.1.** A subset $A$ of a topological space $X$ is called *strongly gsp-open set* if $A^c$ is strongly $gsp$-closed.

**Theorem 4.2.** *A subset $A$ in $X$ is strongly gsp-open in $X$ if and only if $F \subseteq sp\,\text{int}(A)$ whenever $F$ is g-closed and $F \subseteq A$.*

**Proof.** Assume that $A$ is strongly $gsp$-open in $X$. Let $F$ be $g$-closed and $F \subseteq A$. This implies $F^c$ is $g$-open and $A^c \subseteq F^c$. Since $A^c$ is strongly $gsp$-closed, $spcl(A^c) \subseteq F^c$. Since $spcl(A^c) = (sp\,\text{int}(A))^c$, $(sp\,\text{int}(A))^c \subseteq F^c$. Therefore $F \subseteq sp\,\text{int}(A)$.

Conversely, assume that $F \subseteq sp\,\text{int}(A)$ whenever $F$ is $g$-closed and $F \subseteq A$. Let $G$ be a $g$-open set in $X$ containing $A^c$. Therefore $G^c$ is a $g$-closed set contained in $A$. By hypothesis, $G^c \subseteq sp\,\text{int}(A)$. Taking

complements, $G \supseteq spcl(A^c)$. Therefore $A^c$ is strongly *gsp*-closed in $X$. Hence $A$ is strongly *gsp*-open in $X$.

**Remark 4.3.** Intersection of two strongly *gsp*-open sets need not be a strongly *gsp*-open set. In the topological space given in Remark 3.4, the sets $\{b, c\}$ and $\{a, c\}$ are strongly *gsp*-open sets but their intersection $\{c\}$ is not strongly *gsp*-open.

**Theorem 4.4.** *If a set $A$ is strongly gsp-open in $X$, then $G = X$ whenever $G$ is g-open and $sp\,\mathrm{int}(A) \cup A^c \subseteq G$.*

**Proof.** Suppose that $G$ is $g$-open and that $sp\,\mathrm{int}(A) \cup A^c \subseteq G$. Now $G^c \subseteq spcl(A^c) \cap A = spcl(A^c) - A^c$. Since $G^c$ is $g$-closed and $A^c$ is strongly *gsp*-closed, by Theorem 3.10, $G^c = \varnothing$ and hence $G = X$.

**Theorem 4.5.** *If a set $A$ is strongly gsp-closed, then $spcl(A) - A$ is strongly gsp-open.*

**Proof.** Assume that $A$ is strongly *gsp*-closed. By Theorem 3.10, $spcl(A) - A$ contains no nonempty $g$-closed set. Therefore $spcl(A) - A$ is strongly *gsp*-open.

**Theorem 4.6.** *For each $x \in X$, either $\{x\}$ is g-closed or $\{x\}^c$ is strongly gsp-closed.*

**Proof.** If $\{x\}$ is not $g$-closed, then the only $g$-open set containing $\{x\}^c$ is $X$. Thus $spcl(\{x\}^c)$ is contained in $X$ and hence $\{x\}^c$ is strongly *gsp*-closed.

## 5. Strongly Semipre-$T_{1/2}$ Spaces

In this section, we introduce strongly semipre-$T_{1/2}$ spaces and study some of their properties.

**Definition 5.1.** A space $X$ is called *strongly semipre-$T_{1/2}$* if every *gsp*-closed set in $X$ is strongly *gsp*-closed set in $X$.

**Theorem 5.2.** *Every semipre-$T_{1/2}$ space is strongly semipre-$T_{1/2}$ but not conversely.*

**Proof.** Let $X$ be a semipre-$T_{1/2}$. Let $A$ be any $gsp$-closed set in $X$. Since $X$ is a semipre-$T_{1/2}$, $A$ is semi-preclosed in $X$. By Theorem 3.2, $A$ is strongly $gsp$-closed set. Therefore $X$ is strongly semipre-$T_{1/2}$.

The converse of the above theorem need not be true as is seen from the following example.

**Example 5.3.** Consider the topological space $X = \{a, b, c\}$ with topology $\tau = \{X, \varnothing, \{a\}, \{a, c\}\}$. The space $X$ is strongly semipre-$T_{1/2}$ but not semipre-$T_{1/2}$.

**Theorem 5.4.** *Every $T_{1/2}$-space is strongly semipre-$T_{1/2}$ but not conversely.*

**Proof.** Let $X$ be a $T_{1/2}$-space. Every $T_{1/2}$-space is a semipre-$T_{1/2}$ [7, Theorem 4.2]. Using Theorem 5.2, $X$ is strongly semipre-$T_{1/2}$.

The converse of the above theorem need not be true as is seen from the following example.

**Example 5.5.** The space given in Example 5.3 is strongly semipre-$T_{1/2}$ but not $T_{1/2}$.

**Theorem 5.6.** *Every $\alpha$-space is strongly semipre-$T_{1/2}$ but not conversely.*

**Proof.** Let $X$ be an $\alpha$-space. Every $\alpha$-space is a semipre-$T_{1/2}$ [7, Theorem 4.9]. Using Theorem 5.2, $X$ is strongly semipre-$T_{1/2}$.

The converse of the above theorem need not be true as is seen from the following example.

**Example 5.7.** The space given in Example 5.3 is strongly semipre-$T_{1/2}$ but not an $\alpha$-space.

**Theorem 5.8.** *Every singleton set in a strongly semipre-$T_{1/2}$ space $X$ is closed or strongly gsp-open in $X$.*

**Proof.** Assume that for some $x \in X$ the set $\{x\}$ is not closed. Then the only open set containing $\{x\}^c$ is $X$ and hence $\{x\}^c$ is *gsp*-closed. Since $X$ is strongly semipre-$T_{1/2}$, $\{x\}^c$ is strongly *gsp*-closed. Therefore $\{x\}$ is strongly *gsp*-open in $X$.

**Corollary 5.9.** *If $X$ is strongly semipre-$T_{1/2}$, then every singleton set in $X$ is closed or gsp-open.*

**Remark 5.10.** Strongly semipre-$T_{1/2}$ space and semi-$T_{1/2}$ space are independent.

**Example 5.11.** Let $X = \{a, b, c\}$, $\tau = \{X, \varnothing, \{a\}\}$ and $\sigma = \{X, \varnothing, \{a\}, \{b, c\}\}$. The space $(X, \tau)$ is a semi-$T_{1/2}$ but not strongly semipre-$T_{1/2}$ and the space $(X, \sigma)$ is strongly semipre-$T_{1/2}$ but not semi-$T_{1/2}$.

**Remark 5.12.** Strongly semipre-$T_{1/2}$ and $T_d$ are independent.

**Example 5.13.** The space given in Examples 5.11 is $T_d$ but not strongly semipre-$T_{1/2}$. Consider the topological space $X = \{a, b, c\}$ with $\sigma = \{\varnothing, X, \{a\}, \{a, b\}\}$. Then $(X, \sigma)$ is strongly semipre-$T_{1/2}$ but not $T_d$.

We sum up the above results in the following diagram

$$\begin{tikzcd}
 & \text{semipre-}T_{1/2}\text{ space} \arrow[d, shift right] & T_d\text{-space} \arrow[dl, shift right, "/" marking] \\
T_{1/2}\text{-space} \arrow[r, shift left, "/" marking] & \text{strongly semipre-}T_{1/2}\text{ space} \arrow[l, shift left] \arrow[u, shift right, "/" marking] \arrow[ur, shift right, "/" marking] \arrow[r, shift right, "/" marking] \arrow[d, shift right, "/" marking] & \alpha\text{-space} \arrow[l, shift right] \\
 & \text{semi-}T_{1/2}\text{-space} \arrow[u, shift right, "/" marking] &
\end{tikzcd}$$

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# ON A QUESTION POSED BY HUCKABA-PAPICK, IV

RYÛKI MATSUDA

( Received October 20, 2000 )

Submitted by K. K. Azad

## Abstract

This is a simple note for my previous paper [Far East J. Math. Sci. (FJMS) 2(5) (2000), 695-701]. We get some equivalent conditions for a commutative ring $A$ to be a $v$-ring.

This is a simple note or an addendum, and a rappel for my previous paper [6].

A submonoid $S$ of a torsion-free abelian (additive) group is called a *g-monoid*. In this note, a $g$-monoid that is not $\{0\}$ is called a *g-monoid*.

Let $D[S]$ be the semigroup ring of a $g$-monoid $S$ over an integral domain $D$. Let $f = \sum a_i X^{s_i}$ be an element of $D[S]$ where $a_i \neq 0$ for each $i$, and $s_i \neq s_j$ for $i \neq j$. Then the ideal $(s_1, .., s_n)$ of $S$ is denoted by $e(f)$. A non-zerodivisor of a commutative ring $R$ with identity is called a *regular element* of $R$. If an ideal $I$ of $R$ contains a regular element, then $I$ is called a *regular ideal*. If each regular ideal of $R$ is generated by regular elements, then $R$ is called a *Marot ring*. If, for each regular element $f$ of the polynomial ring $R[X]$ of an indeterminate $X$, the ideal $c(f)$ generated

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by the coefficients of $f$ is a regular ideal of $R$, then $R$ is said to have property (A). In the following, $A$ denotes a Marot ring with property (A).

In [6], we showed the following Theorems 1-3.

**Theorem 1.** *Let* $U = \{f \mid f$ *is a regular element of* $A[S]$ *with* $c(f)^{-1} = A\}$, *and assume that* $A[S]_U$ *is a Prüfer ring. Then*

(1) *$A$ is a Prüfer $v$-multiplication ring.*

(2) $A[S]_U = \{f/g \mid f$ *is a non-zero element of* $A[S]$ *and* $g$ *is a regular element of* $A[S]$ *with* $c(f)^v \subset c(g)^v\} \cup \{0\}$.

(3) $A[S]_U$ *is an r-Bezout ring.*

(4) *Each regular prime ideal of* $A[S]_U$ *is the extension of a regular prime ideal of $A$.*

**Theorem 2.** *Let* $U = \{f \mid f$ *is a regular element of* $A[S]$ *with* $c(f)^{-1} = A\}$, *and assume that $A$ is a $v$-ring. Then the following conditions are equivalent*:

(0) $A[S]_U$ *is a Prüfer ring.*

(1) *$A$ is a Prüfer $v$-multiplication ring.*

(2) $A[S]_U$ *coincides with the Kronecker function ring* $A_v^S$ *of $A$ with respect to the $v$-operation on $A$ and $S$.*

(3) $A[S]_U$ *is an r-Bezout ring.*

(4) *Each regular prime ideal of* $A[S]_U$ *is the extension of a prime ideal of $A$.*

**Theorem 3.** *Let* $U = \{f \mid f$ *is a non-zero element of* $D[S]$ *with* $e(f)^{-1} = S\}$, *and assume that $S$ is a $v$-semigroup. Then the following conditions are equivalent*:

(0) $D[S]_U$ *is a Prüfer ring.*

(1) *$S$ is a Prüfer $v$-multiplication semigroup.*

(2) $D[S]_U$ *coincides with the Kronecker function ring* $S_v^D$ *of* $S$ *with respect to the* $v$*-operation on* $S$ *and* $D$.

(3) $D[S]_U$ *is a Bezout ring.*

(4) *Each prime ideal of* $D[S]_U$ *is the extension of a prime ideal of* $S$.

In [6] we posed the question: Is $A$ a $v$-ring in Theorem 1?

Let $F(R)$ be the set of non-zero fractional ideals of a ring $R$, and let $L$ be the total quotient ring of $R$. A mapping $I \mapsto I^*$ of $F(R)$ to $F(R)$ is called a *star-operation* on $R$ if the following conditions hold for all regular elements $a \in L$ and $I, J \in F(R)$:

(1) $(a)^* = (a)$,

(2) $(aI)^* = aI^*$,

(3) $I \subset I^*$,

(4) $I^* \subset J^*$ if $I \subset J$,

(5) $(I^*)^* = I^*$.

If, for $J_1, J_2 \in F(R)$ and a finitely generated regular fractional ideal $I$, $(IJ_1)^* \subset (IJ_2)^*$ implies $J_1^* \subset J_2^*$, then $*$ is called an *a.b. star-operation*. If, for finitely generated $J_1, J_2 \in F(R)$ and a finitely generated regular fractional ideal $I$, $(IJ_1)^* \subset (IJ_2)^*$ implies $J_1^* \subset J_2^*$, then $*$ is called an *e.a.b. star-operation*.

**Lemma 1** (A part of [2, Lemma 4]). *If* $A$ *is a Prüfer* $v$*-multiplication ring, then the* $v$*-operation on* $A$ *is a.b.*

**Proof.** Let $I$ be a finitely generated regular ideal of $A$, and let $J_1, J_2$ be non-zero ideals of $A$ such that $(IJ_1)^v \subset (IJ_2)^v$. It suffices to show that $J_1^v \subset J_2^v$. Let $K$ be the total quotient ring of $A$, and let $x$ be a regular element of $K$ such that $J_2 \subset (x)$. Then we have $(I(J_1, x))^v = I^v(x)$.

There exists a finitely generated regular fractional ideal $J$ such that $(IJ)^v = A$. The equation $J(I(J_1, x))^v = JI^v(x)$ implies that $(J_1, x)^v = (x)$. Hence $J_1^v \subset (x)$. It follows that $J_1^v \subset J_2^v$. Therefore, the $v$-operation on $A$ is a.b.

By Lemma 1, if $A[S]_U$ is a Prüfer ring, then the $v$-operation on $A$ is a.b., hence $A$ is a $v$-ring.

Let $D$ be a domain, and let $E = \{f/g \mid f, g$ are non-zero elements of $D[X]$ such that $c(f)^v \subset c(g)^v\} \cup \{0\}$. If $D$ is a $v$-ring, then $E$ is well-defined. That is, if $f, g, s, t$ are non-zero elements of $D[X]$ such that $c(f)^v \subset c(g)^v$ and that $f/g = s/t$, then $c(s)^v \subset c(t)^v$ (cf. [1, Proof of Theorem (32.7)]). There naturally arises the question: If $E$ is well-defined, then is $D$ a $v$-ring?

**Proposition 1.** *Let $E = \{f/g \mid f$ is a non-zero element of $A[S]$ and $g$ is a regular element of $A[S]$ with $c(f)^v \subset c(g)^v\} \cup \{0\}$. Then $E$ is well-defined if and only if $A$ is a $v$-ring.*

**Proof.** The sufficiency is clear. Assume that $E$ is well-defined. Let $I$, $J_1$, $J_2$ be finitely generated non-zero fractional ideals of $A$ with $I$ regular, and assume that $(IJ_1)^v \subset (IJ_2)^v$. We must show that $J_1^v \subset J_2^v$. We may assume that $I$, $J_1$, $J_2$ are (integral) ideals of $A$. Let $a \in J_1$, and let $x$ be a regular element of the total quotient ring of $A$ such that $J_2 \subset (x)$. Then we have $(Ia)^v \subset (Ix)^v$. There exists a regular element $f$ of $A[S]$ such that $c(f) = I$. We have $c(af)^v \subset c(xf)^v$, and hence $(af)/(xf) \in E$. Since $E$ is well-defined, it follows that $(a)^v \in (x)^v$, hence $a \in J_2^v$, and hence $J_1 \subset J_2^v$. Therefore $J_1^v \subset J_2^v$.

Let $*$ be a star-operation on $A$, and assume that $A$ is a Prüfer $*$-multiplication ring, then $*$ need not be an e.a.b. star-operation ([2, Remark 7]).

Let $F'(R)$ be the set of non-zero $R$-submodules of $L$, where $L$ is the total quotient ring of $R$. A mapping $I \mapsto I^*$ of $F'(R)$ to $F'(R)$ is called a *semistar-operation* on $R$ if the following conditions hold for all regular elements $a$ of $L$ and $I, J \in F'(R)$:

(1) $(aI)^* = aI^*$,

(2) $I \subset I^*$,

(3) $I^* \subset J^*$ if $I \subset J$,

(4) $(I^*)^* = I^*$.

If, for all $J_1, J_2 \in F'(R)$ and a finitely generated regular $I \in F'(R)$, $(IJ_1)^* \subset (IJ_2)^*$ implies $J_1^* \subset J_2^*$, then $*$ is called an *a.b. semistar-operation*. If, for all finitely generated $J_1, J_2 \in F'(R)$ and a finitely generated regular $I \in F'(R)$, $(IJ_1)^* \subset (IJ_2)^*$ implies $J_1^* \subset J_2^*$, then $*$ is called an *e.a.b. semistar-operation*.

The mapping $I \mapsto I^{v'} = (I^{-1})^{-1}$ of $F'(R)$ to $F'(R)$ is a semistar-operation on $R$ called the $v'$*-operation*.

Let $P$ be a prime ideal of a ring $R$. Then the overring $\{x \in L \mid sx \in R$ for some $s \in R - P\}$ of $R$ is denoted by $R_{[P]}$. Let $R'$ be an overring of $R$. If there exists a multiplicative system $T$ in $R$ such that $R' = R_T$ and that $T$ consists of regular elements, then $R'$ is called a *regular quotient ring* of $R$. If $A$ is a Prüfer ring, then $A$ is a Prüfer $*$-multiplication ring for each star-operation $*$.

Theorem 2 can be generalized as follows:

**Theorem 4** [5]. *Let $*$ be an e.a.b. star-operation on $A$, and let $T = \{g \mid g$ is a regular element of $A[S]$ with $c(g)^* = A\}$. Then the following conditions are equivalent:*

(0) *$A[S]_T$ is a Prüfer ring.*

(1) *A is a Prüfer $*$-multiplication ring.*

(2) $A[S]_T = A_*^S$.

(3) $A[S]_T$ *is an r-Bezout ring.*

(4) *Each regular prime ideal of* $A[S]_T$ *is the extension of a prime ideal of A.*

(5) $A_*^S$ *is a regular quotient ring of* $A[S]$.

(6) *Each prime ideal of* $A[S]_T$ *is the contraction of a prime ideal of* $A_*^S$.

(7) *Each regular prime ideal of* $A[S]_T$ *is the contraction of a prime ideal of* $A_*^S$.

(8) *Each valuation overring of* $A_*^S$ *is of the form* $A[S]_{[PA[S]]}$, *where P is a prime ideal of A such that* $A_{[P]}$ *is a valuation overring of A.*

(9) $A_*^S$ *is a flat* $A[S]$*-module.*

Moreover, there exists a Prüfer Marot ring $A$ with property (A) which satisfies the following conditions: Let $*$ be any e.a.b. $*$-operation on $A$. Then there exists a prime ideal of $A[\mathbf{Z}_0]_T$ which is not the extension of a prime ideal of $A$, where $\mathbf{Z}_0$ is the $g$-monoid of non-negative integers.

For the proof of equivalence of (0)-(9) we confer [5, Propositions 3.1 and 3.9 and Theorem 3.7]. Let $k$ be a field, let $X_1, X_2, ..., Y_1, Y_2, ...$ be indeterminates, and let $D_0$ be a Prüfer domain.

Let

$$R = k[[X_1, X_2, ..., Y_1, Y_2, ...]]_1/(X_iX_j,\ Y_iY_j \mid i \neq j),$$

and let $A = R \oplus D_0$, where

$$k[[X_1, X_2, ..., Y_1, Y_2, ...]]_1 = \cup_{n=1}^{\infty} k[[X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n]],$$

and $(X_iX_j, Y_iY_j \mid i \neq j)$ is the ideal of $k[[X_1, X_2, ..., Y_1, Y_2, ...]]_1$ generated by the subset $\{X_iX_j, Y_iY_j \mid i \neq j\}$. Then $A$ is such a ring (cf. [3, Theorem (1.3)]).

A similar result to Theorem 4 holds for semistar-operations on the ring $A$ as follows.

**Theorem 5** [5]. *Let $*$ be an e.a.b. semistar-operation on $A$, and let $W = \{g \mid g$ is a regular element of $A^*[S]$ such that $c(g)^* = A^*\}$. Then the following conditions are equivalent:*

(0) *$A^*[S]_W$ is a Prüfer ring.*

(1) *$A$ is a Prüfer $*$-multiplication ring.*

(2) *$A^*[S]_W$ coincides with the Kronecker function ring $A_*^S$ of $A$ with respect to $*$ and $S$.*

(3) *$A^*[S]_W$ is an r-Bezout ring.*

(4) *Each regular prime ideal of $A^*[S]_W$ is the extension of a prime ideal of $A^*$.*

(5) *$A_*^S$ is a regular quotient ring of $A^*[S]$.*

(6) *Each prime ideal of $A^*[S]_W$ is the contraction of a prime ideal of $A_*^S$.*

(7) *Each regular prime ideal of $A^*[S]_W$ is the contraction of a prime ideal of $A_*^S$.*

(8) *Each valuation overring of $A_*^S$ is of the form $A^*[S]_{[QA^*[S]]}$, where $Q$ is a prime ideal of $A^*$ such that $(A^*)_{[Q]}$ is a valuation overring of $A^*$.*

(9) *$A_*^S$ is a flat $A^*[S]$-module.*

For the proof we confer [5, Propositions 3.2, 3.8 and 3.9].

Finally, Theorem 3 can be generalized as follows:

**Theorem 6** [7]. *Let $D$ be a domain, and let $*$ be an e.a.b. star-operation on a g-monoid $S$, and let $T = \{g \mid g$ is a non-zero element of $D[S]$ with $e(g)^* = S\}$. The following conditions are equivalent:*

(0) *$D[S]_T$ is a Prüfer ring.*

(1) *$S$ is a Prüfer $*$-multiplication semigroup.*

(2) *$D[S]_T$ coincides with the Kronecker function ring $S_*^D$ of $S$ with respect to $*$ and $D$.*

(3) *$D[S]_T$ is a Bezout ring.*

(4) *Each prime ideal of $D[S]_T$ is the extension of a prime ideal of $S$.*

(5) *$S_*^D$ is a quotient ring of $D[S]$.*

(6) *Each prime ideal of $D[S]_T$ is the contraction of a prime ideal of $S_*^D$.*

(7) *Each valuation overring of $S_*^D$ is of the form $D[S]_{PD[S]}$, where $P$ is a prime ideal of $S$ such that $S_P$ is a valuation oversemigroup of $S$.*

(8) *$S_*^D$ is a flat $D[S]$-module.*

For the proof we confer [7, Theorems 8 and 25].

A similar result to Theorem 6 holds for semistar-operations on $S$.

**Theorem 7** [4]. *Let $*$ be an e.a.b. semistar-operation on $S$, and let $W = \{g \mid g$ is a non-zero element of $D[S^*]$ such that $e(g)^* = S^*\}$. The following conditions are equivalent:*

(0) *$D[S^*]_W$ is a Prüfer ring.*

(1) *$S$ is a Prüfer $*$-multiplication semigroup.*

(2) *$D[S^*]_W = S_*^D$.*

(3) *$D[S^*]_W$ is a Bezout ring.*

(4) *Each prime ideal of $D[S^*]_W$ is the extension of a prime ideal of $S^*$.*

(5) *$S_*^D$ is a quotient ring of $D[S^*]$.*

(6) *Each prime ideal of $D[S^*]_W$ is the contraction of a prime ideal of $S_*^D$.*

(7) *Each valuation overring of $S_*^D$ is of the form $D[S^*]_{QD[S^*]}$, where $Q$ is a prime ideal of $S^*$ such that $(S^*)_Q$ is a valuation oversemigroup of $S^*$.*

(8) *$S_*^D$ is a flat $D[S^*]$-module.*

For the proof we confer [4, Proposition 4 and Theorem 23].

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# ON THE GENERALIZED HYERS-ULAM STABILITY OF A QUADRATIC MAPPING

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Submitted by K. K. Azad

**Abstract**

In this paper, we prove a generalization of the stability of the quadratic equation $f(x+y+z)+f(x-y)+f(y-z)+f(x-z)=3f(x)+3f(y)+3f(z)$ in the spirits of Hyers, Ulam, Rassias and Găvruta.

## 1. Introduction

In 1940, S. M. Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems [12]. Among those was the question concerning the stability of homomorphisms:

*Let $G_1$ be a group and let $G_2$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?*

The case of approximately additive mappings was solved by D. H. Hyers [5] under the assumption that $G_1$ and $G_2$ are Banach spaces. In 1978, Th. M. Rassias [10] gave a generalization of the Hyers's result.

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Recently, Găvruta [4] also obtained a further generalization of the Hyers-Ulam-Rassias theorem.

The quadratic functional equation

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0 \tag{1.1}$$

clearly has $f(x)=cx^2$ as a solution with $c$ an arbitrary constant when $f$ is a real function of a real variable. We define any solution of (1.1) to be a *quadratic function*. A Hyers-Ulam stability theorem for the equation (1.1) was proved by F. Skof for functions $f: V \to X$ where $V$ is a normed space and $X$ is a Banach space [11]. In the paper [3], S. Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equation (1.1) and this result was generalized by a number of mathematicians [2, 6, 7, 8, 9].

Throughout this paper, let $V$ be a normed space and $X$ be a Banach space. Consider the following functional equation:

$$f(x+y+z)+f(x-y)+f(y-z)+f(x-z)=3f(x)+3f(y)+3f(z). \tag{1.2}$$

Recently, the first author investigated the Hyers-Ulam stability problem of the equation (1.2) [1]. In Section 2 of this paper, we try to prove the generalized Hyers-Ulam stability of a quadratic functional equation (1.2) and prove the Hyers-Ulam stability of the equation on restricted domains.

## 2. The Results

We denote by $\varphi : (V \setminus \{0\})^3 \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y, z) := \sum_{k=0}^{\infty} \frac{1}{3^{2(k+1)}} \varphi(3^k x, 3^k y, 3^k z) < \infty \tag{a}$$

for all $x, y, z \in V \setminus \{0\}$ or

$$\overline{\varphi}(x, y, z) := \sum_{k=0}^{\infty} 3^{2k} \varphi\left(\frac{x}{3^{k+1}}, \frac{y}{3^{k+1}}, \frac{z}{3^{k+1}}\right) < \infty \tag{a'}$$

for all $x, y, z \in V \setminus \{0\}$.

From now on, we will use the following abbreviation:

$$Df(x, y, z) = f(x + y + z) + f(x - y) + f(y - z) + f(x - z)$$
$$- 3f(x) - 3f(y) - 3f(z).$$

**Theorem 1.** *Let $\varphi$ be as above. Let $V$ be a normed space and $X$ be a Banach space. Suppose that the function $f : V^3 \to X$ satisfies*

$$\| Df(x, y, z) \| \le \varphi(x, y, z) \tag{2.1}$$

*for all $x, y, z \in V \setminus \{0\}$. Then there exists exactly one quadratic function $Q : V \to X$ such that*

$$\left\| Q(x) - f(x) + \frac{3}{8} f(0) \right\| \le \widetilde{\varphi}(x, x, x)$$

*for all $x \in V \setminus \{0\}$. The function $Q$ is given by*

$$Q(x) = \lim_{k \to \infty} \frac{f(3^k x)}{3^{2k}} \quad \text{for all } \ x \in V \tag{2.2}$$

*if $\varphi$ satisfies* (a) *or*

$$Q(x) = \begin{cases} \lim_{k\to\infty} 3^{2k} \left( f\left(\frac{x}{3^k}\right) - \frac{3}{8} f(0) \right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

*if $\varphi$ satisfies* (a′).

**Proof.** We now prove the result for the case $\varphi$ satisfies the condition (a). Replacing $x, y, z$ by $x$ in (2.1), we obtain

$$\left\| f(3x) + 3f(0) - 3^2 f(x) \right\| \le \varphi(x, x, x) \tag{2.3}$$

for $x \in V \setminus \{0\}$. Dividing (2.3) by $3^2$, we obtain

$$\left\| \frac{1}{3^2} \left( f(3x) - \frac{3}{8} f(0) \right) - \left( f(x) - \frac{3}{8} f(0) \right) \right\| \le \frac{1}{3^2} \varphi(x, x, x) \tag{2.4}$$

for $x \in V \setminus \{0\}$. Applying an induction argument on $k$ in (2.4), we easily obtain

$$\left\| \frac{1}{3^{2k}}\left(f(3^k x) - \frac{3}{8}f(0)\right) - \left(f(x) - \frac{3}{8}f(0)\right) \right\| \le \sum_{l=0}^{k-1} \frac{1}{3^{2(l+1)}} \varphi(3^l x,\, 3^l x,\, 3^l x)$$

for $x \in V \setminus \{0\}$ and for all $k \in \mathbb{N}$. This shows that $\left\{\frac{1}{3^{2k}}\left(f(3^k x) - \frac{3}{8}f(0)\right)\right\}$ is a Cauchy sequence. Because $X$ is a Banach space, the sequence $\left\{\frac{1}{3^{2k}}\left(f(3^k x) - \frac{3}{8}f(0)\right)\right\}$ converges. Define $Q : V \to X$ by

$$Q(x) = \lim_{k\to\infty} \frac{f(3^k x) - \frac{3}{8}f(0)}{3^{2k}} = \lim_{k\to\infty} \frac{f(3^k x)}{3^{2k}} \tag{2.5}$$

for all $x \in V$. From (2.5), we easily know that

$$Q(0) = 0 \quad \text{and} \quad Q(3x) = 3^2 Q(x)$$

for all $x \in V$. Replacing $x$, $y$, $z$ by $3^k x$, $3^k y$, $3^k z$, respectively and dividing (2.1) by $3^{2k}$, we obtain

$$\begin{aligned} &\left\| \frac{f(3^k x + 3^k y + 3^k z)}{3^{2k}} + \frac{f(3^k x - 3^k y)}{3^{2k}} + \frac{f(3^k y - 3^k z)}{3^{2k}} \right. \\ &\qquad \left. + \frac{f(3^k x - 3^k z)}{3^{2k}} - 3\frac{f(3^k x)}{3^{2k}} - 3\frac{f(3^k y)}{3^{2k}} - 3\frac{f(3^k z)}{3^{2k}} \right\| \\ &\le \frac{\varphi(3^k x,\, 3^k y,\, 3^k z)}{3^{2k}} \end{aligned} \tag{2.6}$$

for all $x, y, z \in V \setminus \{0\}$. Taking the limit in (2.6) as $k \to \infty$, we get

$$Q(x+y+z) + Q(x-y) + Q(y-z) + Q(x-z) - 3Q(x) - 3Q(y) - 3Q(z) = 0 \tag{2.7}$$

for all $x, y, z \in V \setminus \{0\}$. Replacing $x$ by $y$ and replacing $y$, $z$ by $x$ in (2.7), we obtain

$$Q(y + 2x) + 2Q(y - x) - 6Q(x) - 3Q(y) = 0 \tag{2.8}$$

for all $x, y \in V \setminus \{0\}$. Replacing $z$ by $x$ in (2.7), we obtain

$$Q(y + 2x) + Q(x - y) + Q(y - x) - 6Q(x) - 3Q(y) = 0 \tag{2.9}$$

for all $x, y \in V \setminus \{0\}$. From (2.8) and (2.9), we obtain

$$Q(x) = Q(-x) \tag{2.10}$$

for all $x \in V$. Replacing $y, z$ by $x, -x$, respectively in (2.7) and by (2.10), we obtain

$$Q(x) + 2Q(2x) = 3^2 Q(x) \tag{2.11}$$

for all $x \in V \setminus \{0\}$. From (2.11), we obtain

$$Q(2x) = 4Q(x) \tag{2.12}$$

for all $x \in V$. Replacing $x, y$ by $x + y, x - y$, respectively and replacing $z$ by $x$ in (2.7), we obtain

$$Q(3x) + Q(2y) + Q(y) + Q(-y) = 3Q(x) + 3Q(x + y) + 3Q(x - y) \tag{2.13}$$

for $x, x + y, x - y \in V \setminus \{0\}$. From (2.10), (2.12) and (2.13), we obtain

$$Q(x + y) + Q(x - y) - 2Q(x) - 2Q(y) = 0$$

for $x, x + y, x - y \in V \setminus \{0\}$. Hence we have

$$Q(x + y) + Q(x - y) - 2Q(x) - 2Q(y) = 0$$

for all $x, y \in V$. From (2.4), we have the inequality

$$\left\| Q(x) - f(x) + \frac{3}{8} f(0) \right\| \leq \widetilde{\varphi}(x, x, x)$$

for all $x \in V \setminus \{0\}$. In case $\varphi$ satisfies the condition (a′), we can prove the result by the similar method as in case $\varphi$ satisfies the condition (a).

**Corollary 2.** *Let $p \neq 2$, $\theta > 0$ be real numbers. Suppose that the function $f : V \to X$ satisfies*

$$\| Df(x, y, z) \| \leq \theta(\| x \|^p + \| y \|^p + \| z \|^p)$$

*for all $x, y, z \neq 0$. Then there exists exactly one quadratic function*

*$Q : V \to X$ such that*

$$\left\| Q(x) - f(x) + \frac{3}{8} f(0) \right\| \le \frac{3\theta}{|3^2 - 3^p|} \| x \|^p$$

*for all $x \in V \setminus \{0\}$. The function $Q$ is given by (2.2).*

**Proof.** Since $\varphi(x, y, z) = \theta(\| x \|^p + \| y \|^p + \| z \|^p)$ satisfies the condition (a) or (a′), Theorem 1 says that there exists a unique quadratic function $Q : V \to X$ such that

$$\begin{aligned}\left\| Q(x) - f(x) + \frac{3}{8} f(0) \right\| &\le \sum_{k=0}^{\infty} \frac{1}{3^{2(k+1)}} \varphi(3^k x, 3^k x, 3^k x) \\ &= \sum_{k=0}^{\infty} \frac{\theta}{3^{2(k+1)}} \left( \left\| 3^k x \right\|^p + \left\| 3^k x \right\|^p + \left\| 3^k x \right\|^p \right) \\ &= \frac{\theta}{3} \| x \|^p \sum_{k=0}^{\infty} 3^{(p-2)k} \\ &= \frac{\theta}{3} \| x \|^p \frac{3^2}{|3^2 - 3^p|} \\ &= \frac{3\theta}{|3^2 - 3^p|} \| x \|^p\end{aligned}$$

for all $x \in V \setminus \{0\}$.

The Hyers-Ulam stability of the quadratic equation (1.2) was presented by the first author as the following theorem [1]:

**Corollary 3.** *Suppose that the function $f : V \to X$ satisfies the following inequality*

$$\| Df(x, y, z) \| \le \varepsilon \tag{2.14}$$

*for all $x, y, z \in V$, then there exists a unique quadratic function $Q : V \to X$ such that the inequality*

$$\| f(x) - Q(x) \| \leq \frac{4}{5}\varepsilon$$

*holds for all* $x \in V$. *If, moreover, f is measurable or* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in V$, *then the function* $Q$ *satisfies* $Q(tx) = t^2 Q(x)$ *for all* $x \in V$ *and* $t \in \mathbb{R}$.

Similarly, the Hyers-Ulam stability of the quadratic equation (1.2) on an unbounded domain is obtained as the following:

**Theorem 4.** *Let* $d > 0$ *and* $\varepsilon > 0$ *be given. If a function* $f : V \to X$ *satisfies the inequality* (2.14) *for all* $x, y, z \in V$ *with* $\|x\| + \|y\| + \|z\| \geq d$, *then there exists a unique quadratic function* $Q : V \to X$ *such that*

$$\left\| Q(x) - f(x) - \frac{5}{2} f(0) \right\| \leq \frac{62}{5}\varepsilon \tag{2.15}$$

*for all* $x \in V$. *If, moreover, f is measurable or* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in V$, *then the function* $Q$ *satisfies* $Q(tx) = t^2 Q(x)$ *for all* $x \in V$ *and* $t \in \mathbb{R}$.

**Proof.** Suppose that $\| x \| + \| y \| + \| z \| < d$. Choose a $t \in V$ with $\| t \| \geq 2d$. Then

$$\begin{aligned} &\| x \| + \| y - t \| + \| z + t \| \geq d; \qquad \| x - y \| + \| t \| + \| t \| \geq d; \\ &\| x \| + \| z \| + \| t \| \geq d; \qquad \| z + t \| + \| x \| + \| x \| \geq d; \\ &\| x \| + \| t \| + \| t \| \geq d; \qquad \| y \| + \| z \| + \| t \| \geq d. \end{aligned} \tag{2.16}$$

From (2.14), (2.16) and the following relation

$$\begin{aligned} &\| 2f(x + y + z) + 2f(x - y) + 2f(y - z) + 2f(x - z) \\ &\quad - 6f(x) - 6f(y) - 6f(z) + 5f(0) \| \\ &\leq \| f(x + y + z) + f(x - y + t) + f(y - z - 2t) + f(x - z - t) \\ &\quad - 3f(x) - 3f(y - t) - 3f(z + t) \| \end{aligned}$$

$$
\begin{aligned}
&+\| f(x+y+z)+f(x-y-t)+f(y-z-t)+f(x-z-2t) \\
&\qquad -3f(x-t)-3f(y)-3f(z+t)\| \\
&+\| -f(x-y)-f(x-y+t)-f(-2t)-f(x-y-t) \\
&\qquad +3f(x-y)+3f(-t)+3f(t)\| \\
&+\| -f(y-z)-f(y-z+t)-f(-2t)-f(y-z-t) \\
&\qquad +3f(y-z)+3f(-t)+3f(t)\| \\
&+\| -f(x-z)-f(x-z+t)-f(-2t)-f(x-z-t) \\
&\qquad +3f(x-z)+3f(-t)+3f(t)\| \\
&+\| -f(y+z)-f(-y+t)-f(y-z-2t)-f(-z-t) \\
&\qquad +3f(0)+3f(y-t)+3f(z+t)\| \\
&+\| -f(x+z)-f(-x+t)-f(x-z-2t)-f(-z-t) \\
&\qquad +3f(0)+3f(x-t)+3f(z+t)\| \\
&+\| f(x-z+t)+f(x+z)+f(-z-t)+f(x-t) \\
&\qquad -3f(x)-3f(-z)-3f(t)\| \\
&+\| f(y-z+t)+f(y+z)+f(-z-t)+f(y-t) \\
&\qquad -3f(y)-3f(-z)-3f(t)\| \\
&+\| -f(-x+t)-f(x-t)-f(-x+t)-f(0) \\
&\qquad +3f(0)+3f(-x+t)+3f(0)\| \\
&+\| -f(-y+t)-f(y-t)-f(-y+t)-f(0) \\
&\qquad +3f(0)+3f(-y+t)+3f(0)\| \\
&+3\| -f(-z+t)-f(z-t)-f(-z+t)-f(0) \\
&\qquad +3f(0)+3f(-z+t)+3f(0)\|
\end{aligned}
$$

$$+ \| f(0) + f(-2t) + f(t) + f(-t) - 3f(-t) - 3f(t) - 3f(0) \|$$

$$+ 2\| f(-2t) + f(0) + f(-t) + f(-t) - 3f(-t) - 3f(-t) - 3f(0) \|$$

$$+ 3\| f(z + t) + f(z) + f(-t) + f(z - t) - 3f(z) - 3f(0) - 3f(t) \|$$

$$+ 3\| - f(-z + t) - f(z + t) - f(-z) - f(t)$$

$$+ 3f(t) + 3f(-z) + 3f(0) \|$$

$$+ 2\| - f(0) - f(2t) - f(-t) - f(t) + 3f(t) + 3f(-t) + 3f(0) \|$$

$$+ 2\| f(0) + f(t) + f(t) + f(2t) - 3f(t) - 3f(0) - 3f(-t) \|$$

$$+ 4\| f(t) + f(t) + f(0) + f(t) - 3f(t) - 3f(0) - 3f(0) \|$$

$$\leq 31\varepsilon$$

we obtain

$$\| Df(x, y, z) \| \leq \frac{1}{2} \| 2f(x + y + z) + 2f(x - y) + 2f(y - z) + 2f(x - z)$$

$$- 6f(x) - 6f(y) - 6f(z) + 5f(0) \| + \left\| \frac{5}{2} f(0) \right\|$$

$$\leq \frac{1}{2} \cdot 31\varepsilon + \left\| \frac{5}{2} f(0) \right\|.$$

Hence the inequality holds for all $x, y, z \in V$. Therefore, the assertions of our theorem are immediate consequence of Corollary 2.

Define $S = \{(x, y, z) \in V^3 : \| x \| < d, \| y \| < d, \| z \| < d\}$ for some given $d > 0$. The fact $\{(x, y, z) \in V^3 : \| x \| + \| y \| + \| z \| \geq 3d\} \subset V^3 \setminus S$ implies that the following corollaries are consequences of Theorem 4.

**Corollary 5.** *If a function $f : V \to X$ satisfies the inequality* (2.14) *for all $(x, y, z) \in V^3 \setminus S$, then there exists a unique quadratic function $Q : V \to X$ which satisfies the inequality* (2.15) *for all $x \in V$. Moreover, if $f$ is measurable or $f(tx)$ is continuous in $t \in \mathrm{R}$ for each fixed $x \in V$, then the function $Q$ satisfies $Q(tx) = t^2 Q(x)$ for all $x \in V$ and $t \in \mathrm{R}$.*

**Corollary 6.** *A function $f : V \to X$ is a quadratic function if and only if* $\| Df(x, y, z) \| \to 0$ *as* $\| x \| + \| y \| + \| z \| \to \infty$.

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# A NOTE ON ABSOLUTE RIESZ SUMMABILITY FACTORS

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Submitted by K. K. Azad

## Abstract

H. Bör [Kuwait J. Sci. Engg. 23 (1996), 1-5] has proved a theorem which deals with $|\overline{N}, p_n|_k$ summability factors of infinite series. This generalizes an earlier result due to him. But the result fails for $p_n = \frac{1}{n}$ for all $n \in N$. In this article, we generalize this result.

## 1. Introduction

Throughout this article sums without limit means the summation is from $n = 1$ to $\infty$. Let $\sum a_n$ be a given infinite series and $(s_n)$ be its sequence of partial sums. Let $(t_n)$ denote the $n^{\text{th}}$ $(C, 1)$ mean of $(na_n)$. Then the series $\sum a_n$ is said to be *summable* $|C, 1|_k$, $k \geq 1$ if

$$\sum_n \frac{|t_n|^k}{n} < \infty. \tag{1.1}$$

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Let $(p_n)$ be a positive sequence and

$$P_n = \sum_{r=0}^{n} p_r \to \infty \quad \text{as } n \to \infty.$$

Let $(u_n)$ be the $(\overline{N}, p_n)$ mean of $(s_n)$. Then we have

$$u_n = \frac{1}{P_n} \sum_{r=0}^{n} p_r s_r.$$

The series $\sum a_n$ is said to be *summable* $|\overline{N}, p_n|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |u_n - u_{n-1}|^k < \infty. \tag{1.2}$$

Obviously, $|\overline{N}, 1|_k = |C, 1|_k$.

Bör [3] proved the following result.

**Theorem A.** *Let $(X_n)$ be a positive non-decreasing sequence and the sequences $(\beta_n)$ and $(\lambda_n)$ be such that*

$$|\Delta\lambda_n| \leq \beta_n, \quad \textit{where } \Delta\lambda_n = \lambda_n - \lambda_{n+1} \tag{1.3}$$

$$\beta_n \to 0 \quad \textit{as } n \to \infty \tag{1.4}$$

$$\lambda_n X_n = O(1) \quad \textit{as } n \to \infty \tag{1.5}$$

$$\sum_{n=1}^{\infty} nX_n |\Delta\beta_n| < \infty \tag{1.6}$$

$$P_n = O(np_n) \tag{1.7}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad \textit{as } m \to \infty. \tag{1.8}$$

*Then the series $\sum a_n\lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$.*

The above result is an improvement of the following results due to Bör [3].

**Theorem B** [1]. *Let $(p_n)$ be a sequence of positive numbers such that (1.7) holds. Let $(X_n)$ be a positive non-decreasing sequence and let there exist a sequence $(\lambda_n)$ such that*

$$\sum_{n=1}^{\infty} nX_n | \Delta^2\lambda_n | < \infty. \tag{1.9}$$

*If the conditions (1.5) and (1.8) of Theorem A hold, then the series $\sum a_n\lambda_n$ is summable $| \overline{N}, p_n |_k$, $k \geq 1$.*

**Theorem C** [2]. *Let $(X_n)$ be a positive non-decreasing sequence and $(p_n)$ be a sequence of positive numbers such that (1.7) holds. Let $(\lambda_n)$ be a sequence such that*

$$\lambda_n \to 0 \quad as\ n \to \infty. \tag{1.10}$$

*If (1.8) and (1.9) hold, then the series $\sum a_n\lambda_n$ is summable $| \overline{N}, p_n |_k$, $k \geq 1$.*

## 2. Main Result

It is clear that the condition (1.7) of Theorem A fails if we take $p_n = \dfrac{1}{n+1}$, for all $n = 0, 1, 2, 3, \ldots$. In order to improve Theorem A, we prove the following result.

**Theorem.** *Let $(X_n)$ be a positive non-decreasing sequence and there exist sequences $(\beta_n)$ and $(\lambda_n)$ such that*

$$\sum_{n=1}^{\infty} \frac{| \lambda_n |}{n} < \infty \tag{2.1}$$

$$\sum_{n=1}^{m} \frac{1}{n} | t_n |^k = O(X_m) \quad as\ m \to \infty. \tag{2.2}$$

*If the conditions (1.3), (1.4), (1.5), (1.6) and (1.8) of Theorem A hold, then the series $\sum a_n\lambda_n$ is summable $| \overline{N}, p_n |_k$, $k \geq 1$.*

The following lemma will be used for establishing the result.

**Lemma** [5]. *If $(X_n)$ is a positive non-decreasing sequence and $(\beta_n)$ is a positive sequence such that* (1.4) *and* (1.6) *hold, then*

$$n\beta_n X_n \to 0 \quad as \quad n \to \infty \tag{2.3}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{2.4}$$

**Proof of the Theorem.** Let $(T_n)$ denote the $n^{\text{th}}$ $(\overline{N}, p_n)$ mean of $\sum a_n\lambda_n$. Then

$$T_n = \frac{1}{P_n}\sum_{r=0}^{n} p_r \sum_{i=1}^{r} a_i\lambda_i$$

$$= \frac{1}{P_n}\sum_{r=0}^{n}(P_n - P_{r-1})\, a_r\lambda_r.$$

For $n \ge 1$, we have

$$T_n - T_{n-1} = \frac{(n+1)\,p_n\lambda_n t_n}{nP_n} - \frac{p_n}{P_nP_{n-1}}\sum_{r=1}^{n-1} p_r\lambda_r\,\frac{r+1}{r}\,t_r$$

$$+ \frac{p_n}{P_nP_{n-1}}\sum_{r=1}^{n-1} P_r\Delta\lambda_r\,\frac{r+1}{r}\,t_r + \frac{p_n}{P_nP_{n-1}}\sum_{r=1}^{n-1} P_r\,\frac{\lambda_{r+1}}{r}\,t_r$$

$$= T_{n,\,1} + T_{n,\,2} + T_{n,\,3} + T_{n,\,4}, \text{ say.}$$

Since

$$\left|\sum_{i=1}^{4} T_{n,\,i}\right|^k \le 4^{k-1}\sum_{i=1}^{4}\left|\,T_{n,\,i}\,\right|^k.$$

(One may refer to problem 2, page 22 of Maddox [4]), to complete the proof it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\,T_{n,\,i}\,\right|^k < \infty \quad \text{for } i = 1, 2, 3, 4.$$

The proof for $i = 1, 2$ follows from the proof of Bör [3]. Now by (1.3) and Hölder's inequality, we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,3}\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{r=1}^{n-1}P_r\beta_r|t_r|^k\right\}\left\{\sum_{r=1}^{n-1}P_r\beta_r\right\}^{k-1}$$

$$= O(1)\sum_{r=1}^{m}\beta_r|t_r|^k \qquad \text{by (2.4) and since } P_n \text{ is increasing}$$

$$= O(1)\sum_{r=1}^{m-1}|r\Delta\beta_r|\sum_{i=1}^{r}\frac{|t_i|^k}{i} + O(1)m\beta_m\sum_{r=1}^{m}\frac{|t_r|^k}{r}$$

by Abel's summation formula

$$= O(1)\sum_{r=1}^{m-1}r|\Delta\beta_r|X_r + O(1)\sum_{r=1}^{m-1}\beta_{r+1}X_r + O(1)m\beta_mX_m$$

$$= O(1). \qquad \text{by (1.6), (2.3) and (2.4)}$$

Again by Hölder's inequality, we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,4}\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{r=1}^{n-1}|t_r|^kP_r\frac{|\lambda_{r+1}|}{r}\right\}\left\{\sum_{r=1}^{n-1}P_r\frac{|\lambda_{r+1}|}{r}\right\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{r=1}^{n-1}|t_r|^kP_r\frac{|\lambda_{r+1}|}{r}\right\} \qquad \text{by (2.1)}$$

$$= O(1)\sum_{r=1}^{m}|\lambda_{r+1}|\frac{|t_r|^k}{r}$$

$$= O(1)\sum_{r=1}^{m-1} \beta_{r+1}X_{r+1} + O(1)|\lambda_{m+1}|\, X_{m+1}$$

by Abel's summation formula and (2.2)

$$= O(1)$$

by (1.5) and (2.4).

This completes the proof.

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# SOME PROPERTIES OF r-$T_0$ AND r-$T_1$ SPACES

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## Abstract

We define r-quasi-$T_0$, r-sub $T_0$, r-$T_0$ and r-$T_1$ spaces in smooth fuzzy topological spaces and investigate some of their properties. Moreover, we study their subspaces and products.

## 1. Introduction and Preliminaries

In [1, 5, 8, 9, 15, 16], various separation axioms in fuzzy topological spaces were introduced. R. Srivastava [16] introduced separation axioms in a view of the definition of R. N. Hazra et al. [7].

In this paper, we define r-quasi-$T_0$, r-sub $T_0$, r-$T_0$ and r-$T_1$ spaces in smooth fuzzy topological spaces in a view of the definition of A. P. Sostak [14]. We investigate some of their properties. Also, we study their subspaces and products.

Throughout this paper, let $X$ be a nonempty set, $I = [0, 1]$ and $I_0 = (0, 1]$. For $\alpha \in I$, $\tilde{\alpha}(x) = \alpha$ for all $x \in X$. A *fuzzy point* $x_t$ for $t \in I_0$ is an element of $I^X$ such that

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$$x_t(y) = \begin{cases} t & \text{if } y = x, \\ 0 & \text{if } y \in X - \{x\}. \end{cases}$$

The set of all fuzzy points in $X$ is denoted by $Pt(X)$. A fuzzy point $x_t \in \lambda$ iff $t \leq \lambda(x)$. A fuzzy set $\lambda$ is *quasi-coincident* with $\mu$, denoted by $\lambda\ q\ \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. If $\lambda$ is not quasi-coincident with $\mu$, we denote $\lambda\, \overline{q}\, \mu$.

**Lemma 1.1** [12]. *Let $f : X \to Y$ be a function. For $\lambda, \mu, \nu, \mu_i \in I^X$ and $\rho \in I^Y$, we have the following properties*:

(1) *If $\lambda\ q\ \mu$ and $\mu \leq \nu$, then $\lambda\ q\ \nu$.*

(2) *$\lambda \leq \mu$ iff $x_t\ q\ \lambda$ implies $x_t\ q\ \mu$ iff $x_t \in \lambda$ implies $x_t \in \mu$.*

(3) *$x_t\ q \bigvee_{i \in \Gamma} \mu_i$ iff there exists $j \in \Gamma$ such that $x_t\ q\ \mu_j$.*

(4) *If $\lambda\, q\, \mu$, then $f(\lambda)\, q\, f(\mu)$.*

(5) *$\lambda\, q\, f^{-1}(\rho)$ iff $f(\lambda)\, q\, \rho$.*

**Definition 1.2** [3, 14]. A function $\tau : I^X \to I$ is called a *smooth fuzzy topology* on $X$ if it satisfies the following conditions:

(O1) $\tau(\widetilde{0}) = \tau(\widetilde{1}) = 1$.

(O2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$ for each $\mu_1, \mu_2 \in I^X$.

(O3) $\tau(\bigvee_{i \in \Gamma} \mu_i) \geq \bigwedge_{i \in \Gamma} \tau(\mu_i)$ for any $\{\mu_i\}_{i \in \Gamma} \subset I^X$.

The pair $(X, \tau)$ is called a *smooth fuzzy topological space*. A smooth fuzzy topological space $(X, \tau)$ is called *stratified* if

(S) $\tau(\widetilde{\alpha}) = 1$ for each $\alpha \in I$.

Let $\tau_1$ and $\tau_2$ be smooth fuzzy topologies on $X$. We say $\tau_1$ is *finer* than $\tau_2$ ($\tau_2$ is *coarser* than $\tau_1$) if $\tau_2(\mu) \leq \tau_1(\mu)$ for all $\mu \in I^X$.

**Theorem 1.3** [3]. *Let* $(X, \tau)$ *be a fuzzy topological space. For each* $r \in I_0$, $\lambda \in I^X$, *define an operator* $C_\tau : I^X \times I_0 \to I^X$ *as follows:*

$$C_\tau(\lambda, r) = \bigwedge\{\mu \mid \mu \geq \lambda,\ \tau(\widetilde{1} - \mu) \geq r\}.$$

*Then it satisfies the following properties:*

(1) $C_\tau(\widetilde{0}, r) = \widetilde{0}$, $C_\tau(\widetilde{1}, r) = \widetilde{1}$, *for all* $r \in I_0$.

(2) $C_\tau(\lambda, r) \geq \lambda$.

(3) $C_\tau(\lambda_1, r) \leq C_\tau(\lambda_2, r)$, *if* $\lambda_1 \leq \lambda_2$.

(4) $C_\tau(\lambda \vee \mu, r) = C_\tau(\lambda, r) \vee C_\tau(\mu, r)$, *for all* $r \in I_0$.

(5) $C_\tau(\lambda, r) \leq C_\tau(\lambda, r')$, *if* $r \leq r'$, *where* $r, r' \in I_0$.

(6) $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

**Definition 1.4** [10]. Let $\Theta_X$ be a subset of $I^X$ which does not contain $\widetilde{0}$. A function $\beta : \Theta_X \to I$ is called a *smooth fuzzy topological base* on $X$ if it satisfies the following conditions:

(B1) $\beta(\widetilde{1}) = 1$.

(B2) $\beta(\mu_1 \wedge \mu_2) \geq \beta(\mu_1) \wedge \beta(\mu_2)$, for all $\mu_1, \mu_2 \in \Theta_X$.

A smooth fuzzy topological base $\beta$ always *generates* a smooth fuzzy topology $\tau_\beta$ on $X$ in the following sense:

**Theorem 1.5** [10]. *Let* $\beta$ *be a smooth fuzzy topological base on X. For each* $\mu \in I^X$, *define the function* $\tau_\beta : I^X \to I$ *as follows:*

$$\tau_\beta(\mu) = \begin{cases} \bigvee\left\{\bigwedge_{i \in J} \beta(\mu_i)\right\} & \text{if } \mu = \bigvee_{i \in J} \mu_i, \quad \mu_i \in \Theta_X, \\ 1 & \text{if } \mu = \widetilde{0}, \\ 0 & \text{otherwise,} \end{cases}$$

*where the first* $\bigvee$ *is taken over all families* $\{\mu_i \in \Theta_X \mid \mu = \bigvee_{i \in J} \mu_i\}$. *Then* $(X, \tau_\beta)$ *is a smooth fuzzy topological space.*

**Definition 1.6** [10]. If $\beta$ is a smooth fuzzy topological base on $X$, then $\tau_\beta$ is called the *smooth fuzzy topology generated by* $\beta$. The pair $(X, \tau_\beta)$ is called a *smooth fuzzy topological space generated by a base* $\beta$ on $X$.

**Definition 1.7.** Let $(X, \tau_1)$ and $(Y, \tau_2)$ be smooth fuzzy topological spaces and $f : X \to Y$ be a function. Then

(1) $f$ is called *fuzzy continuous* if $\tau_2(\mu) \leq \tau_1(f^{-1}(\mu))$ for all $\mu \in I^Y$.

(2) $f$ is called *fuzzy open* if $\tau_1(\lambda) \leq \tau_2(f(\lambda))$ for all $\lambda \in I^X$.

(3) $f$ is called a *fuzzy homeomorphism* if $f$ is bijective fuzzy continuous and $f^{-1}$ is fuzzy continuous.

**Theorem 1.8** [10, 13]. *Let* $(X_i, \tau_i)_{i \in \Gamma}$ *be smooth fuzzy topological spaces,* $X$ *be a set and* $f_i : X \to X_i$ *be a function, for each* $i \in \Gamma$. *Let* $\Theta_X = \{\widetilde{0} \neq \mu = \bigwedge_{i \in F} f_i^{-1}(\nu_i) \mid \tau_i(\nu_i) > 0,\ i \in F\}$ *be given, for every finite index set* $F \subset \Gamma$. *Define a function* $\beta : \Theta_X \to I$ *on* $X$ *by*

$$\beta(\mu) = \bigvee \left\{ \bigwedge_{i \in F} \tau_i(\nu_i) \mid \mu = \bigwedge_{i \in F} f_i^{-1}(\nu_i) \right\},$$

*where the first* $\bigvee$ *is taken over all finite index subset* $F$ *of* $\Gamma$. *Then*

(1) $\beta$ *is a smooth fuzzy topological base on* $X$.

(2) *The smooth fuzzy topology* $\tau_\beta$ *generated by* $\beta$ *is the coarsest smooth fuzzy topology on* $X$ *for which each* $i \in \Gamma$, $f_i$ *is fuzzy continuous.*

(3) *A map* $f : (Z, \tau_Z) \to (X, \tau_\beta)$ *is fuzzy continuous iff for each* $i \in \Gamma$, $f_i \circ f$ *is fuzzy continuous.*

From Theorem 1.8, we can have the following definitions.

**Definition 1.9** [10]. Let $(X, \tau)$ be a smooth fuzzy topological space

and $A$ be a subset of $X$. The pair $(A, \tau|_A)$ is said to be a *subspace* of $(X, \tau)$ if $\tau|_A$ is the coarsest smooth fuzzy topology on $A$ for which the inclusion map $i$ is fuzzy continuous.

**Definition 1.10** [10]. Let $X$ be the product $\prod_{i\in\Gamma} X_i$ of the family $\{(X_i, \tau_i) \mid i \in \Gamma\}$ of smooth fuzzy topological spaces. Then the coarsest smooth fuzzy topology $\tau = \otimes\tau_i$ on $X$ for which each the projections $\pi_i : X \to X_i$ is fuzzy continuous is called the *product smooth fuzzy topology* of $\{\tau_i \mid i \in \Gamma\}$, and $(X, \tau)$ is called the *product smooth fuzzy topological space*.

**Theorem 1.11** [11]. *Let $(X, \tau)$ be a product space of a family $\{(X_i, \tau_i) \mid i \in \Gamma\}$ of smooth fuzzy topological spaces and $(X_j, \tau_j)$ be stratified. Then for every slice $\widetilde{X}_j$ in $X$ parallel to $X_j$, $\pi_j|_{\widetilde{X}_j} : \widetilde{X}_j \to X_j$ is a fuzzy homeomorphism.*

**Definition 1.12** [4]. Let $(X, \tau)$ be smooth fuzzy topological space. For $\mu \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$, we denote

$$\mathcal{Q}_\tau(x_t, r) = \{\mu \in I^X \mid x_t\, q\, \mu,\ \tau(\mu) \geq r\}.$$

A fuzzy set $\mu \in \mathcal{Q}_\tau(x_t, r)$ is called an *r-Q open neighborhood* of $x_t$.

## 2. r-quasi-$T_0$, r-sub $T_0$, r-$T_0$ and r-$T_1$ Spaces

From Definition 1.12, we can have the following:

**Definition 2.1.** A smooth fuzzy topological space $(X, \tau)$ is said to be

(1) *r-quasi-$T_0$-space* if for each $x_t, x_s \in Pt(X)$ and $t < s$, there exists $\lambda \in Q_\tau(x_s, r)$ such that $x_t\, \overline{q}\, \lambda$.

(2) *r-sub $T_0$-space* if for each $x \neq y \in X$, there exists $t \in I_0$ such that there exists $\lambda \in Q_\tau(x_t, r)$ such that $y_t\, \overline{q}\, \lambda$, or there exists $\mu \in Q_\tau(y_t, r)$ such that $y_t\, \overline{q}\, \mu$.

(1) $(X, \tau)$ *is an* $r$-$T_1$ *space.*

(2) *For each* $x_t \in Pt(X)$, $x_t = C_\tau(x_t, r)$.

(3) *For each* $\lambda \in I^X$, $\lambda = \bigwedge\{\mu \mid \lambda \le \mu,\ \tau(\mu) \ge r\}$.

**Proof.** (1) $\Rightarrow$ (2) We only show that $C_\tau(x_t, r) \le x_t$. Let $y_s \in C_\tau(x_t, r)$. Suppose that $y_s \not\le x_t$. Since $(X, \tau)$ is an r-$T_1$ space, there exists $\lambda \in Q_\tau(y_s, r)$ such that $x_t \bar{q} \lambda$. It implies $x_t \le \tilde{1} - \lambda$ with $\tau(\lambda) \ge r$. Hence $C_\tau(x_t, r) \le \tilde{1} - \lambda$. Since $y_s \in C_\tau(x_t, r) \le \tilde{1} - \lambda$, we have $\lambda \notin Q_\tau(y_s, r)$. It is a contradiction. Hence $y_s \le x_t$. Thus $y_s \in C_\tau(x_t, r)$ implies $y_s \le x_t$. By Lemma 1.1(2), $C_\tau(x_t, r) \le x_t$.

(2) $\Rightarrow$ (3) Let $\rho = \bigwedge\{\mu \mid \lambda \le \mu,\ \tau(\mu) \ge r\}$. We only show that $\rho \le \lambda$. Suppose there exist $x \in X$ and $t \in I_0$ such that

$$\rho(x) > 1 - t \ge \lambda(x). \tag{A}$$

Then $\lambda \le \tilde{1} - x_t$. Since $x_t = C_\tau(x_t, r)$,

$$\tau(\tilde{1} - x_t) = \tau(\tilde{1} - C_\tau(x_t, r)) \ge r.$$

Hence $\rho \le \tilde{1} - x_t$. It is a contradiction for the equation (A).

(3) $\Rightarrow$ (1) For each $x_t, y_s \in Pt(X)$ such that $x_t \not\le y_s$, $\tilde{1} - x_t \not\ge \tilde{1} - y_s$. From (3), since $\tilde{1} - y_s = \bigwedge\{\mu \mid \tilde{1} - y_s \le \mu,\ \tau(\mu) \ge r\}$, there exists $\mu \in I^X$ such that

$$\tilde{1} - y_s \le \mu, \quad \tau(\mu) \ge r.$$

Moreover, since $\tilde{1} - x_t \not\ge \tilde{1} - y_s$, we have $x_t \, q \, \tilde{1} - y_s$. Since $\tilde{1} - y_s \le \mu$, by Lemma 1.1(1), we have $x_t \, q \, \mu$. Thus $\mu \in Q_\tau(x_t, r)$ such that $y_s \bar{q} \mu$. Hence $(X, \tau)$ is an r-$T_1$ space.

**Theorem 2.7.** *Let* $(X, \tau)$ *be a stratified smooth fuzzy topological space. Then* $(X, \tau)$ *is an* $r$*-quasi-*$T_0$ *space for all* $r \in I_0$.

**Proof.** Let $x_t,\ x_s \in Pt(X)$ such that $t < s$. Then there exists $\alpha \in I_0$ such that

$$t \le 1 - \alpha < s.$$

Since $(X, \tau)$ is a stratified smooth fuzzy topological space, we have $\tau(\widetilde{\alpha}) = 1$. Hence $\widetilde{\alpha} \in Q_\tau(x_s, r)$ such that $x_t\ \overline{q}\ \widetilde{\alpha}$.

The following theorem is easily proved.

**Theorem 2.8.** (1) *Every r-$T_0$ space is both r-quasi-$T_0$ and r-sub $T_0$.*

(2) *Every r-$T_1$ space is r-$T_0$.*

The converse of Theorem 2.8(1) is not true follows from the following example.

**Example 2.9.** Let $X = \{x, y\}$ be a set. We define a smooth fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \widetilde{\alpha} \text{ for } \alpha \in I, \\ \dfrac{1}{3}, & \text{if } \lambda = \mu_{pq}, \\ 0, & \text{otherwise;} \end{cases}$$

where for each $0 < p \le 0.4,\ \mu_{pq}(x) = p,\ \mu_{pq}(y) = q,\ 0 \le q < p$.

Since $(X, \tau)$ is a stratified smooth fuzzy topological space, by Theorem 2.7, $(X, \tau)$ is an r-quasi-$T_0$-space for each $r \in I_0$.

If $r > \dfrac{1}{3}$ and $t \in I_0$, then for each $x_t,\ y_t \in Pt(X)$, we have

$$Q_\tau(x_t, r) = Q_\tau(y_t, r) = \{\widetilde{\alpha} \mid 1 - t < \alpha \le 1\}.$$

By Theorem 2.2(2) and Corollary 2.4(2), $(X, \tau)$ is neither r-sub $T_0$ nor r-$T_0$ for each $r > \dfrac{1}{3}$.

If $0 < r \le \frac{1}{3}$ and $x \neq y \in X$, there exists $0.7 \in I_0$ such that there exists $x_{0.4} = \mu_{\frac{2}{5}0} \in Q_\tau(x_{0.7}, r)$ with $y_{0.7}\, \overline{q}\, \mu_{\frac{2}{5}0} = x_{0.4}$. Hence $(X, \tau)$ is r-sub $T_0$ for each $0 < r \le \frac{1}{3}$.

For $x_{0.3}, y_{0.3} \in Pt(X)$ and $0 < r \le \frac{1}{3}$, we have $Q_\tau(x_{0.3}, r) = Q_\tau(y_{0.3}, r) = \{\widetilde{\alpha} \mid 0.7 < \alpha\}$. Hence it is not r-$T_0$ for $0 < r \le \frac{1}{3}$.

For $0 < r \le \frac{1}{3}$, $(X, \tau)$ is both r-quasi-$T_0$ and r-sub $T_0$, but not r-$T_0$.

The converse of Theorem 2.8(2) is not true follows from the following example.

**Example 2.10.** Let $X = \{x, y\}$ be a set. We define a smooth fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \widetilde{\alpha} \text{ for } \alpha \in I, \\ \frac{1}{2}, & \text{if } \lambda = \mu_{pq}, \\ 0, & \text{otherwise,} \end{cases}$$

where for each $0 < p < 1$, $\mu_{pq}(x) = p$, $\mu_{pq}(y) = q$, $0 \le q < p$.

Let $z_t, z_s \in Pt(X)$ with $t \neq s$ for $z = x$ or $y$. We have

$$Q_\tau\left(z_t, \frac{1}{2}\right) \neq Q_\tau\left(z_s, \frac{1}{2}\right).$$

For each $x_t, y_s \in Pt(X)$, for $p > 1 - t$ we have $\mu_{p0} \in Q_\tau\left(x_t, \frac{1}{2}\right)$ with $y_s\, \overline{q}\, \mu_{p0}$. Hence $(X, \tau)$ is a $\frac{1}{2}$-$T_0$ space.

On the other hand, let $y_{0.5} \not\leq x_{0.5}$. For each $\mu_{pq} \in Q_\tau\left(y_{0.5}, \frac{1}{2}\right)$, since $q + 0.5 > 1$ and $p > q$, we have $x_{0.5}\, q\, \mu_{pq}$, that is, $Q_\tau\left(y_{0.5}, \frac{1}{2}\right) \subset Q_\tau\left(x_{0.5}, \frac{1}{2}\right)$. Thus $(X, \tau)$ is not a $\frac{1}{2}$ - $T_1$ space.

**Theorem 2.11.** *Every subspace of r-quasi-$T_0$ (r-sub $T_0$, r-$T_0$ and r-$T_1$) spaces is r-quasi-$T_0$ (r-sub $T_0$, r-$T_0$ and r-$T_1$, respectively).*

**Proof.** Let $(X, \tau)$ be an r-$T_1$. Let $a_t, b_s \in Pt(A)$ such that $a_t \not\leq b_s$. Then $a_t, b_s \in Pt(X)$ such that $a_t \not\leq b_s$. Since $(X, \tau)$ is r-$T_1$, there exists $\lambda \in Q_\tau(a_t, r)$ such that $b_s\, \overline{q}\, \lambda$. Since $\tau_A(i^{-1}(\lambda)) \geq \tau(\lambda) \geq r$ from Theorem 1.8, we have $i^{-1}(\lambda) \in Q_{\tau|_A}(a_t, r)$ such that $b_s\, \overline{q}\, i^{-1}(\lambda)$. Others are similarly proved.

We can prove the following theorem in a similar method as Theorem 2.11.

**Theorem 2.12.** *Every fuzzy homeomorphic space of r-quasi-$T_0$ (r-sub $T_0$, r-$T_0$ and r-$T_1$) space is r-quasi-$T_0$ (r-sub $T_0$, r-$T_0$ and r-$T_1$, respectively).*

**Theorem 2.13.** *Let $\{(X_i, \tau_i) \mid i \in \Gamma\}$ be a family of r-$T_1$ (r-quasi-$T_0$, r-sub $T_0$ and r-$T_0$) spaces. Let $\tau$ be the product smooth fuzzy topology on $X = \prod_{i \in \Gamma} X_i$. Then $(X, \tau)$ is r-$T_1$ (r-quasi-$T_0$, r-sub $T_0$ and r-$T_0$, respectively).*

**Proof.** Let $x_t, y_s \in Pt(X)$ such that $x_t \not\leq y_s$. Then there exists $i \in \Gamma$ such that $(\pi_i(x))_t \not\leq (\pi_i(y))_s$. Since $(X_i, \tau_i)$ is an r-$T_1$ space, there exists $\lambda \in I^{X_i}$ such that

$$\lambda \in Q_{\tau_i}((\pi_i(x))_t, r), (\pi_i(y))_s\, \overline{q}\, \lambda.$$

Since $\pi_i(x_t) = (\pi_i(x))_t\, q\, \lambda$ iff $x_t\, q\, \pi_i^{-1}(\lambda)$ from Lemma 1.1(5) and

$\tau(\pi_i^{-1}(\lambda)) \geq \tau_i(\lambda)$ from Theorem 1.8, we have

$$\pi_i^{-1}(\lambda) \in Q_\tau(x_t, r),\ y_s\, \overline{q}\, \pi_i^{-1}(\lambda).$$

Therefore, $(X, \tau)$ is an r-$T_1$ space. Others are similarly proved.

**Theorem 2.14.** *Let* $\{(X_i, \tau_i) \mid i \in \Gamma\}$ *be a family of smooth fuzzy topological spaces. Let* $\tau$ *be the product smooth fuzzy topology on* $X = \prod_{i \in \Gamma} X_i$. *If* $(X, \tau)$ *is an* $r$*-sub* $T_0$ *space, then* $(X_i, \tau_i)$ *is an* $(r - \varepsilon)$*-sub* $T_0$ *space for each* $\varepsilon > 0$ *and for each* $i \in \Gamma$.

**Proof.** Let $x^j, y^j \in X_j$ such that $x^j \neq y^j$. Then there exist $x^i \in X_i$ for all $i \in \Gamma - \{j\}$ such that $x \neq y \in X$ and

$$\pi_i(x) = \begin{cases} x^i, & \text{if } i \in \Gamma - \{j\}, \\ x^j, & \text{if } i = j, \end{cases}$$

$$\pi_i(y) = \begin{cases} x^i, & \text{if } i \in \Gamma - \{j\}, \\ y^j, & \text{if } i = j. \end{cases}$$

Since $(X, \tau)$ is an r-sub $T_0$ space, there exists $t \in I_0$ such that

$$\rho \in Q_\tau(x_t, r),\ \ y_t\, \overline{q}\, \rho.$$

Let $\beta$ be a base for $\tau$. Since $\tau(\rho) \geq r$, by Theorem 1.5, for $\varepsilon > 0$, there exists a family $\{\rho_k \mid \rho = \bigvee_{k \in \Lambda} \rho_k\}$ such that

$$\tau(\rho) \geq \bigwedge_{k \in \Lambda} \beta(\rho_k) > r - \varepsilon.$$

Since $x_t\, q(\rho = \bigvee_{k \in \Lambda} \rho_k)$, by Lemma 1.1(3), there exists $k \in \Gamma$ such that $x_t\, q\, \rho_k$ and $\beta(\rho_k) > r - \varepsilon$. From Theorem 1.8, there exists a family $\{\lambda_i \mid \rho_k = \bigwedge_{i \in F} \pi_i^{-1}(\lambda_i)\}$ where $F$ is a finite subset of $\Gamma$ such that

$$\beta(\rho_k) \geq \bigwedge_{i \in F} \tau_i(\lambda_i) > r - \varepsilon. \tag{1}$$

Without loss of generality, we may assume $j \in F$ because we can take $F_1 = F \cup \{j\}$ such that $\lambda_j = \widetilde{1}$, $\tau_j(\widetilde{1}) = 1$, if necessary. Since $x_t \, q \, \rho_k$ and $y_t \, \overline{q} \, \rho_k$,

$$t > \bigvee_{i \in F - \{j\}} (\widetilde{1} - \lambda_i)(\pi_i(x)) \vee (\widetilde{1} - \lambda_j)(x^j), \tag{II}$$

$$t \leq \bigvee_{i \in F - \{j\}} (\widetilde{1} - \lambda_i)(\pi_i(x)) \vee (\widetilde{1} - \lambda_j)(y^j). \tag{III}$$

If $\bigvee_{i \in F - \{j\}} (\widetilde{1} - \lambda_i)(\pi_i(x)) \geq t$, it is a contradiction for equations (II) and (III). Thus

$$\bigvee_{i \in F - \{j\}} (\widetilde{1} - \lambda_i)(\pi_i(x)) < t.$$

It implies

$$t > (\widetilde{1} - \lambda_j)(x^j), \quad t \leq (\widetilde{1} - \lambda_j)(y^j).$$

Furthermore, by the equation (I), we have $\tau_j(\lambda_j) > r - \varepsilon$. Hence

$$\lambda_j \in Q_{\tau_j}((x^j)_t, r - \varepsilon), \quad (y^j)_t \, \overline{q} \, \lambda_j.$$

Thus, $(X_j, \tau_j)$ is an $(r - \varepsilon)$-sub $T_0$ space.

In the above theorem, if $(X, \tau)$ is r-$T_1$ (r-quasi-$T_0$ and r-$T_0$, respectively), the result may not hold is shown by the following example.

**Example 2.15.** Let $X = \{x\}$, $Y = \{y\}$ be sets. We define smooth fuzzy topologies $\tau_1 : I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda = \widetilde{\alpha} \text{ for } \alpha \in I, \\ 0, & \text{otherwise} \end{cases}$$

and $\tau_2 : I^Y \to I$ as follows:

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda = \widetilde{0} \text{ or } \widetilde{1}, \\ \dfrac{1}{2}, & \text{if } \lambda = y_{0.2}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $X \times Y = \{(x, y)\}$ be a product set and $\tau_1 \otimes \tau_2$ the product smooth fuzzy topology on $X \times Y$. Since $(x, y)_{0.2} = \pi_1^{-1}(\widetilde{0.2}) = \pi_2^{-1}(y_{0.2})$, by Theorem 1.8, we have

$$\tau_1 \otimes \tau_2(\widetilde{0.2}) = \tau_1(\widetilde{0.2}) \vee \tau_2(y_{0.2}) = 1.$$

We can obtain the product smooth fuzzy topology $\tau_1 \otimes \tau_2 : I^{X \times Y} \to I$ as follows:

$$\tau_1 \otimes \tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda = \widetilde{\alpha} \text{ for } \alpha \in I, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(X \times Y, \tau_1 \otimes \tau_2)$ are r-$T_1$, r-$T_0$ and r-quasi-$T_0$ for all $r \in I_0$. But $(Y, \tau_2)$ is not r-quasi-$T_0$ for all $r \in I_0$. Hence it is neither r-$T_0$ nor r-$T_1$ for all $r \in I_0$.

**Theorem 2.16.** *Let $\{(X_i, \tau_i) \mid i \in \Gamma\}$ be a family of smooth fuzzy topological spaces and $\tau$ be the product smooth fuzzy topology on $X = \prod_{i \in \Gamma} X_i$. If $(X, \tau)$ is r-quasi-$T_0$ (r-sub $T_0$, r-$T_0$, r-$T_1$) and $(X_j, \tau_j)$ is stratified for $j \in \Gamma$, then $(X_j, \tau_j)$ is r-quasi-$T_0$ (r-sub $T_0$, r-$T_0$, r-$T_1$, respectively).*

**Proof.** Let $\widetilde{X}_j$ be the slice of $(X, \tau)$ parallel to $X_j$. Since $\left(\widetilde{X}_j, \tau|_{\widetilde{X}_j}\right)$ is a subspace of $(X, \tau)$, by Theorem 2.11, $\left(\widetilde{X}_j, \tau|_{\widetilde{X}_j}\right)$ is r-quasi-$T_0$ (r-sub $T_0$, r-$T_0$, and r-$T_1$). Since $(X_j, \tau_j)$ is stratified, by Theorem 1.11,

$\pi_j|_{\widetilde{X}_j} : \left(\widetilde{X}_j, \tau|_{\widetilde{X}_j}\right) \to (X_j, \tau_j)$ is a homeomorphism. From Theorem 2.12, $(X_j, \tau_j)$ is r-quasi-$T_0$ (r-sub $T_0$, r-$T_0$ and r-$T_1$, respectively).

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# COMMUTATORS OF SINGULAR INTEGRALS ON $H_b^p(R^n)$ AT CRITICAL INDEX

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## Abstract

We consider the commutator of δ-Calderón-Zygmund singular integral operator and multiplication operator by $b$, that is, the commutator of Coifman, Rochberg and Weiss. We give a counterexample which shows that this commutator is not a bounded operator from $H_b^p(R^n)$ to weak-$L^p(R^n)$ where $p = n/(n+\delta)$.

## 1. Introduction

Let $b$ be a locally integrable function on $R^n$ and let $T$ be a δ-Calderón-Zygmund singular integral operator (see Section 2). Consider the commutator operator $[b, T]$ defined by

$$[b, T]f = b \cdot Tf - T(bf).$$

Coifman, Rochberg and Weiss [2] proved that $[b, T]$ is a bounded operator on $L^p(R^n)$, $1 < p < \infty$, when $b$ is a $\mathrm{BMO}(R^n)$ function. But this operator is not weak type (1, 1) (see [4, p. 175]), and not bounded from $H^p(R^n)$ to $L^p(R^n)$ where $p \leq 1$.

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Key words and phrases: commutator, singular integral, Hardy space.

Pérez [4] and Alvarez [1] defined the space $H_b^p(R^n)$ which is a subspace of the usual $H^p(R^n)$, and showed that $[b, T]$ is a bounded operator from $H_b^p(R^n)$ to $L^p(R^n)$ where $n/(n+\delta) < p \leq 1$.

In this paper, we obtain a counterexample showing that $[b, T]$ is not a bounded operator from $H_b^p(R^n)$ to weak-$L^p(R^n)$ where $p = n/(n+\delta)$.

## 2. Known Results

The following notations are used: For a set $E \subset R^n$ we denote the characteristic function of $E$ by $\chi_E$ and $|E|$ is the Lebesgue measure of $E$.

We say a function $f$ is in weak-$L^p(R^n)$ if

$$\| f \|_{L^{p,\infty}} = \sup_{t>0} t \cdot |\{x \in R^n ; |f(x)| > t\}|^{1/p} < \infty.$$

**Definition 1.** We say $T$ is a $\delta$-Calderón-Zygmund singular integral operator $(0 < \delta \leq 1)$ if $T$ satisfies the following conditions (see [3, p. 118]).

$$T \text{ is a bounded operator on } L^2(R^n). \tag{1}$$

$$Tf(x) = p.v. \int_{R^n} K(x-y) f(y)\, dy. \tag{2}$$

$$|K(x)| \leq C \frac{1}{|x|^n}, \quad x \neq 0. \tag{3}$$

$$|K(x-y) - K(x)| \leq C \frac{|y|^\delta}{|x|^{n+\delta}}, \quad |x| > 2|y|. \tag{4}$$

**Definition 2.** Let $n/(n+1) < p \leq 1$. A function $a$ is an $H_b^p$-atom if there exists a ball $Q$ such that the following conditions are satisfied

$$\operatorname{supp}(a) \subset Q, \tag{5}$$

$$\| a \|_{L^\infty} \leq |Q|^{-1/p}, \tag{6}$$

$$\int_Q a(x)\,dx = 0, \tag{7}$$

$$\int_Q a(x)\,b(x)\,dx = 0. \tag{8}$$

**Definition 3** ([1] and [4]). Let $n/(n+1) < p \leq 1$. A temperate distribution $f$ is said to *belong to* $H_b^p(R^n)$ if, in the $S'$ sense, it can be written as $f = \sum_j \lambda_j a_j$, where $a_j$ are $H_b^p$-atoms and $\sum_j |\lambda_j|^p < \infty$. As usual, we define on $H_b^p(R^n)$ the quasi norm,

$$\| f \|_{H_b^p} = \inf\left(\sum_j |\lambda_j|^p\right)^{1/p},$$

where the infimum is taken over all representations of $f$.

**Remark.** If $p = 1$, then $H_b^1$ is the subspace of $L^1$ and $\|\cdot\|_{H_b^1}$ is norm.

Pérez [4] and Alvarez [1] obtained the next theorem.

**Theorem** (Pérez [4] and Alvarez [1]). *Let $T$ be a $\delta$-Calderón-Zygmund singular integral operator $(0 < \delta \leq 1)$ and let $b$ be a function in* BMO$(R^n)$. *Then $[b, T]$ is a bounded operator from $H_b^p(R^n)$ to $L^p(R^n)$, where $n/(n+\delta) < p \leq 1$.*

**Remark.** Pérez [4] proved for $p = 1$ and Alvarez [1] proved for $p < 1$.

Alvarez [1] proposed the following problem.

**Problem.** Let $T$ be a $\delta$-Calderón-Zygmund singular integral operator $(0 < \delta < 1)$ and let $b$ be a function in BMO$(R^n)$. Whether $[b, T]$ is a bounded operator from $H_b^p(R^n)$ to weak-$L^p(R^n)$, where $p = n/(n+\delta)$.

And Alvarez [1] proved the following:

**Theorem** (Alvarez [1]). *Let $T$ be a $\delta$-Calderón-Zygmund singular integral operator* $(0 < \delta < 1)$ *and let $b$ be a function in* $L^\infty(R^n)$. *Then* $[b, T]$ *is a bounded operator from* $H_b^p(R^n)$ *to weak-*$L^p(R^n)$, *where* $p = n/(n+\delta)$.

### 3. Counterexample

We give a counterexample.

**Counterexample.** *There exists a $\delta$-Calderón-Zygmund singular integral operator* $(0 < \delta < 1)$ *and functions* $b \in \mathrm{BMO}(R^n)$, $a \in H_b^p(R^n)$ *such that* $[b, T]a$ *does not belong to weak-*$L^p(R^n)$, *where* $p = n/(n+\delta)$.

**Proof.** For simplicity we shall show for $n = 1$.

Let

$$\phi(x) = \begin{cases} x^\delta, & 0 \le x \le 1/2, \\ (1-x)^\delta, & 1/2 < x \le 1, \\ 0, & \text{otherwise}. \end{cases}$$

And let $I_j^k = [2^j + 2k, 2^j + 2k + 1]$, where $j = 1, 2, 3, \ldots$, and $k$ is an integer such that $0 \le k \le 2^{j-1} - 1$.

For $x \ge 0$, we define $K(x)$ as

$$K(x) = \begin{cases} 2^{-j(1+\delta)}\phi(x - 2^j - 2k), & \text{if } x \in I_j^k \text{ for some } j, k, \\ 0, & \text{otherwise}. \end{cases}$$

And for $x \le 0$, let $K(x) = -K(-x)$.

We define

$$Tf(x) = \int_{R^1} K(x-y) f(y)\, dy.$$

It is clear that $T$ is a $\delta$-Calderón-Zygmund operator.

We shall show that $[b, T]a$ does not belong to weak-$L^p(R^1)$ for some $b \in \mathrm{BMO}$ and $a \in H_b^p$ where $p = \dfrac{1}{1+\delta}$.

Let $b(x) = \log_2 |x| \cdot \chi_{\{|x| \geq 1\}}$ and

$$a(x) = \begin{cases} 1 & 0 \leq x < 1/2, \\ -1, & 1/2 \leq x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Because $[b, T]a(x) = b(x)Ta(x)$, we show $b \cdot Ta$ does not belong to weak-$L^p$.

Let $I_j^{k*} = [2^j + 2k + 1/4,\ 2^j + 2k + 1/2]$.

For $x \in I_j^{k*}$, we have

$$\begin{aligned} Ta(x) &= 2^{-j(1+\delta)} \int_{2^j+2k}^{x} (y - 2^j - 2k)^{\delta}\, dy \\ &= 2^{-j(1+\delta)} (x - 2^j - 2k)^{\delta+1}/(\delta+1) \\ &\geq C_\delta \cdot 2^{-j(1+\delta)}, \end{aligned}$$

where $C_\delta = 4^{-\delta-1}/(\delta+1)$.

Let $I_j = \cup_k I_j^{k*}$. Then we have $|I_j| = 4^{-1}2^{j-1}$ and $b(x)Ta(x) \geq C_\delta \cdot j \cdot 2^{-j(1+\delta)}$, where $x \in I_j$.

For $n \in \mathbb{N}$, we have

$$\{x;\ b(x)Ta(x) > C_\delta \cdot n \cdot 2^{-n(1+\delta)}\} \supset I_{n-1}$$

and

$$|\{x;\ b(x)Ta(x) > C_\delta \cdot n \cdot 2^{-n(1+\delta)}\}| \geq 4^{-1}2^{n-2} = 16^{-1}2^n.$$

So we obtain

$$\| b \cdot Ta \|_{L^{p,\infty}}^{p} \geq \sup_{n}((C_\delta \cdot n \cdot 2^{-n(1+\delta)})^p \,|\{x;\, b(x)\,Ta(x) > C_\delta \cdot n \cdot 2^{-n(1+\delta)}\}|)$$

$$\geq \sup_{n}(C_\delta^p \cdot n^p \cdot 2^{-n(1+\delta)\,p} \cdot 16^{-1} 2^{n}) = 16^{-1} C_\delta^p \lim_{n\to\infty} n^p = \infty.$$

**Remark.** Similarly, we can give counterexamples for $n \geq 2$.

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# ON THE INTERSECTION AND THE EXTENDIBILITY OF $P_t$-SETS

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## Abstract

For a fixed integer $t$, a $P_t$-set of size $n$ is a set $S = \{x_1, x_2, \ldots, x_n\}$ of distinct positive integers such that $x_i x_j + t$ is a square of an integer, whenever $i \neq j$. In this paper, among other results, we give a method for constructing $P_t$-sets whose intersection contains at least three elements and we study the extendibility of a particular set which is a $P_t$-set for two distinct values of $t$.

For a fixed integer $t$, a $P_t$-set of size $n$ is a set $S = \{x_1, x_2, \ldots, x_n\}$ of distinct positive integers such that $x_i x_j + t$ is a square of an integer, whenever $i \neq j$ (we say that the set $S$ satisfies the Diophantus property $D(t)$). Old examples of $P_t$-sets include the $P_{256}$-set $\{1, 33, 68, 105\}$ found by Diophantus and the $P_1$-set $\{1, 3, 8, 120\}$ found by Fermat. A $P_t$-set $S$ is said *extendible* if there exists an integer $y \notin S$ such that $S \cup \{y\}$ is still a $P_t$-set. The problem of extending $P_t$-sets is very old and date from the time of Diophantus. (For the history of the problem see Dickson [4].)

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Baker and Davenport [2] made a very spectacular advance in this area. Using diophantine approximations and very deep calculations, they showed that the $P_1$-set $\{1, 3, 8, 120\}$ is non-extendible. Several other people later have made efforts to characterize the extendibility of some families of $P_t$-sets (see the references).

Euler generalised Fermat's example and gave a general construction of $P_1$-sets of size four (see [1] and [10]). Unfortunately, there is no general formula for extending $P_t$-sets of size three when $t \neq 1$. In fact, it was proved ([3], [11], [14]) that some $P_t$-sets of size three are non-extendible. A natural question arise:

Is it possible to find a $P_t$-set $S$ and a $P_{t'}$-set $S'$ (with $t \neq t'$) whose intersection contains at least three elements $x$, $y$ and $z$, and such that the set $\{x, y, z\}$ is non-extendible as a $P_t$-set but extendible as a $P_{t'}$-set?

This paper deals with this kind of question. We prove in Theorem 1 that there exists an infinite family of couples $(t, t')$ for which there exist a $P_t$-set $S$ and a $P_{t'}$-set $S'$ such that $S \cap S'$ contains at least three elements. In Theorems 2 and 3, we study the extendibility of a particular $P_t$-set for two distinct values of $t$. In an unpublished notes, Kevin Brown considered sets of three integers in which the product of any two elements increased by one is the double of a square of an integer. The only example he found is the set $\{49, 79, 493\}$. In Theorem 4, we prove that there exist infinitely many such sets.

**Lemma 1.** *Let $x$, $y$ and $z$ any three integers such that $x + y + z$ is even. Then, the set $\{x, y, z\}$ is a $P_t$-set for* $t = \left(\frac{x + y - z}{2}\right)^2 - xy.$

**Lemma 2.** *Let $x$, $y$ and $z$ three positive integers such that $y$ is even and $z + 15x - 10y = 0$. Then, the set $\{x, y, z\}$ is a $P_{t'}$-set for*

$$t' = \left(\frac{x + 4y - z}{4}\right)^2 - xy.$$

The proofs of Lemmas 1 and 2 are straightforward verifications and will be omitted.

**Theorem 1.** *There exist infinitely many couples* $(t, t')$ *for which there exist a* $P_t$*-set and a* $P_{t'}$*-set whose intersection contains at least three elements.*

**Proof.** Let $x$, $y$ and $z$ three positive integers such that $y$ is even and $z + 15x - 10y = 0$. From Lemmas 1 and 2, the set $\{x, y, z\}$ is a $P_t$-set for $t = \left(\frac{x + y - z}{2}\right)^2 - xy$ and a $P_{t'}$-set for $t' = \left(\frac{x + 4y - z}{4}\right)^2 - xy$. It is easy to verify that under the above conditions, $t \neq t'$ iff $y \neq 2x$.

**Some Examples:**

(1) Let $x = 1$ and $y = 2$, then $z = 5$ and $t = t' = -1$. Brown [3] and independently the second author [11], proved that the $P_{-1}$-set $\{1, 2, 5\}$ is non-extendible.

(2) Let $x = 2$ and $y = 6$, then $z = 30$, $t = 109$ and $t' = -11$. Then, the set $\{2, 6, 30\}$ is a $P_{109}$-set and a $P_{-11}$-set.

A small computation gives that the set $\{2, 6, 10, 30\}$ is still a $P_{-11}$-set and we have the following theorem:

**Theorem 2.** *The* $P_{-11}$*-set* $\{2, 6, 10, 30\}$ *is non-extendible.*

**Proof.** Suppose that there exists an integer $a$ such that $\{2, 6, 10, 30, a\}$ is still a $P_{-11}$-set. Then the following system of equations

$$\begin{cases} 2a - 11 = x^2, \\ 6a - 11 = y^2, \\ 10a - 11 = z^2, \\ 30a - 11 = w^2, \end{cases} \tag{1}$$

has an integral solution $(x, y, z, w) \in \mathbb{N}^4$. The system (1) yields

$$\begin{cases} y^2 - 3x^2 = 22 \\ w^2 - 3z^2 = 22. \end{cases} \tag{2}$$

Since $a > 10$, the second equation and the fourth equation in the system (1) gives

$$\sqrt{5}y < w < \sqrt{6}y. \tag{3}$$

The Pellian equation

$$u^2 - 3v^2 = 22$$

has two classes of solution (see [15]) given by

$$\begin{cases} u_n + \sqrt{3}v_n = (5 + \sqrt{3})(2 + \sqrt{3})^n \\ \overline{u}_n + \sqrt{3}\,\overline{v}_n = (5 - \sqrt{3})(2 + \sqrt{3})^n \end{cases} \tag{4}$$

and satisfying the relations

$$\begin{cases} u_n = 4u_{n-1} - u_{n-2}, \\ \overline{u}_n = 4\overline{u}_{n-1} - \overline{u}_{n-2}. \end{cases}$$

Hence, the sequences $u_n$ and $\overline{u}_n$ are increasing and $u_n > 3u_{n-1}$ and $\overline{u}_n > 3\overline{u}_{n-1}$. If $y$ and $w$ in the system (2) are in the same class, then $w > 3y$, which is a contradiction with the inequality (3). From the system (4), we deduce

$$\begin{cases} u_n = \dfrac{(5 + \sqrt{3})(2 + \sqrt{3})^n + (5 - \sqrt{3})(2 - \sqrt{3})^n}{2}, \\ \overline{u}_n = \dfrac{(5 - \sqrt{3})(2 + \sqrt{3})^n + (5 + \sqrt{3})(2 - \sqrt{3})^n}{2} \end{cases}$$

and then we have the following inequalities:

$$\begin{cases} \sqrt{5}\,\overline{u}_n > u_n > \overline{u}_n > 3\overline{u}_{n-1} > \sqrt{6}\,\overline{u}_{n-1}, \\ \sqrt{5}\,u_n > \overline{u}_{n+1} > 3u_{n-1} > \sqrt{6}\,u_{n-1}. \end{cases} \tag{5}$$

The inequalities in (3) and (5) show that $y$ and $w$ in the system (3) could not be in distinct classes. Then, the system (1) has no solution and the $P_{-11}$-set $\{2, 6, 10, 30\}$ is non-extendible.

**Theorem 3.** *The $P_{109}$-set $\{2, 6, 30\}$ is non-extendible.*

**Proof.** Suppose that there exists an integer $a$ such that $\{2, 6, 30, a\}$ is still a $P_{109}$-set. Then the following system of equations

$$\begin{cases} 2a + 109 = A^2, \\ 6a + 109 = B^2, \\ 30a + 109 = C^2, \end{cases} \tag{6}$$

has an integral solution $(A, B, C) \in \mathbb{N}^3$. The system (6) yields

$$(2a + 109)(6a + 109)(30a + 109) = (ABC)^2.$$

Consider the elliptic curve $E$ given by the equation

$$y^2 = (2a + 109)(6a + 109)(30a + 109).$$

From Lutz-Nagell theorem, one obtain that the torsion sub-group of $E(\mathbf{Q})$ is a bi-cyclic group of order 4. One can use apecs or just apply the algorithm of 2-descent of Tate (the 2-isogeny is not trivial) to see that the rank of $E$ is zero. It is easy to verify that the torsion points of $E$ do not give any solution of the system (6).

In an unpublished notes, Kevin Brown considered triples of integers $x$, $y$ and $z$ such that

$$\begin{cases} xy + 1 = 2A^2, \\ xz + 1 = 2B^2, \\ yz + 1 = 2C^2, \end{cases} \tag{7}$$

has an integral solution $(A, B, C)$.

These triples seem to be more rare than the $P_1$-sets. The only example he found is the triple (49, 79, 493). We prove the following:

**Theorem 4.** *There exist infinitely many triples $(x, y, z)$ such that $xy + 1 = 2A^2$, $xz + 1 = 2B^2$ and $yz + 1 = 2C^2$ with $(A, B, C) \in \mathbb{N}^3$.*

**Proof.** Let $a$ and $b$ be two integers satisfying the equation $(7a+b)^2 - 18b^2 = 1$. One can show that there is infinitely many such integers $a$ and $b$. Put $x = a+b$, $y = 17b - 31a$ and $z = 35b - 49a$. It is an easy calculation to verify that $xy + 1 = 2(3a)^2$, $xz + 1 = 2(3b)^2$ and $yz + 1 = 2(17b - 28a)^2$.

**Examples.**

(1) Let $a = -3$ and $b = 4$. Then, $x = 1$, $y = 161$ and $z = 287$.

(2) Let $a = 63$ and $b = 136$. Then, $x = 199$, $y = 359$ and $z = 1673$.

## Concluding Remarks

(1) One can construct a set of integers $\{x, y, z\}$ which is a $P_t$-set for three distinct values of $t$. For example, the set $\{10, 22, 70\}$ is a $P_t$-set for $t = -204$, $t = -171$ and $t = 161$. It would be interesting to know what is the largest possible value of $k$ such that the intersection of $k$ distinct $P_t$-sets contains at least three elements.

(2) Is there a $P_1$-set $S$ of size at least three which is also a $P_t$-set for $t \neq 1$? We conjecture that the answer is no.

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# AN OBSTRUCTION TO TOTALLY REAL SUBMANIFOLDS IN LOCALLY CONFORMAL KÄHLER SPACE FORMS

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## Abstract

In this article, we establish sharp inequalities involving δ-invariants for a totally real submanifold in a locally conformal Kähler space form of constant holomorphic sectional curvature with arbitrary codimension.

## 1. Introduction

Riemannian invariants are the intrinsic characteristics of the Riemannian manifold. Among all Riemannian invariants, curvature is "the $N^{\circ}1$Riemannian invariant and the most natural" according to M. Berger in [1]. Classically, among the Riemannian curvature invariants, geometers have been studying sectional, scalar and Ricci curvature.

Recently, in [3] B. Y. Chen introduced new types of curvature invariants, defining two strings of scalar-valued Riemannian curvature functions, namely $\delta(n_1, \ldots, n_k)$ and $\hat{\delta}(n_1, \ldots, n_k)$ for every $(n_1, \ldots, n_k)$

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satisfying $n_1 < n$, $n_j \geq 2$ and $n_1 + \cdots + n_k \leq n$. For these two strings of Riemannian curvature invariants, one always has trivially $\delta(n_1, \ldots, n_k) \geq \hat{\delta}(n_1, \ldots, n_k)$. We simply called these invariants the *δ-invariants*. The first string of δ-invariants, $\delta(n_1, \ldots, n_k)$, extend naturally the Riemannian invariant introduced in [2]. In [3] B. Y. Chen studied δ-invariants for submanifolds in Riemannian space forms with arbitrary codimension. Also, in [8] A. Oiaga and I. Mihai investigated δ-invariants for slant submanifolds in complex space forms.

In this paper, we study submanifolds of locally conformal Kähler space forms of constant holomorphic sectional curvature with arbitrary codimension and establish δ-invariants for totally real submanifolds in locally conformal Kähler space forms.

## 2. Preliminaries

Let $\widetilde{M}$ be a Hermitian manifold with almost complex structure $J$ and a Hermitian metric $g$. A Hermitian manifold $\widetilde{M}$ is called a *locally conformal Kähler manifold* if each point $p \in \widetilde{M}$ has an open neighborhood $U$ with a differentiable map $\phi : U \to \mathbb{R}$ such that

$$g^* = e^{-2\phi} g|_U \tag{2.1}$$

is Kähler metric on $U$ (see [5, 9]). On the other hand, the fundamental 2-form $w$ of $\widetilde{M}$ is defined by

$$w(X, Y) = g(JX, Y) \tag{2.2}$$

for any tangent vectors $X$, $Y$ on $\widetilde{M}$.

**Proposition 2.1** [5]. *A Hermitian manifold $\widetilde{M}$ is a locally conformal Kähler manifold if and only if there exists a global closed 1-form $\alpha$ satisfying*

$$\begin{aligned}(\widetilde{\nabla}_Z w)(X, Y) &= \beta(Y) g(X, Z) - \beta(X) g(Y, Z) \\ &\quad + \alpha(Y) w(X, Z) - \alpha(X) w(Y, Z)\end{aligned} \tag{2.3}$$

*for any tangent vectors X, Y, Z on* $\widetilde{M}$, *where* $\widetilde{\nabla}$ *denotes the Levi-Civita connection with respect to* $g$ *and the* 1*-form* $\beta$ *is given by* $\beta(X) = -\alpha(JX)$.

The 1-form $\alpha$ is called *Lee form* and its dual vector field is *Lee vector field*. A locally conformal Kähler manifold having parallel Lee form is said to be a *generalized Hopf manifold*. On a locally conformal Kähler manifold, a symmetric (0, 2)-tensor $P$ is defined by

$$P(X, Y) = -(\widetilde{\nabla}_X\alpha)Y - \alpha(X)\alpha(Y) + \frac{1}{2}\|\alpha\|^2 g(X, Y), \tag{2.4}$$

and another (0, 2)-tensor $\widetilde{P}$ by $\widetilde{P}(X, Y) = P(JX, Y)$, where $\|\alpha\|$ is the norm of $\alpha$ with respect to $g$. Denote by $\widetilde{\rho}$ the trace of $P$. We remark that $\widetilde{\rho}$ is a constant on a generalized Hopf manifold.

Let $M$ be an $n$-dimensional submanifold of an $m$-dimensional locally conformal Kähler manifold $\widetilde{M}$. Let $\nabla$ be the induced Levi-Civita connection of $M$. Then the Gauss and Weingarten formulas given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\widetilde{\nabla}_X V = -A_V X + D_X V \tag{2.6}$$

for vector fields $X$, $Y$ tangent to $M$ and a vector field $V$ normal to $M$, where $h$ denotes the second fundamental form, $D$ the normal connection and $A_V$ the shape operator in the direction of $V$. The second fundamental form and the shape operator are related by

$$g(h(X, Y), V) = g(A_V X, Y). \tag{2.7}$$

We also use $g$ for the induced Riemannian metric on $M$ as well as the locally conformal Kähler manifold $\widetilde{M}$. Moreover, the mean curvature vector $H$ on $M$ is defined by $H = \dfrac{1}{n}\, trace\, h$.

For an $n$-dimensional Riemannian manifold $M$, we denote by $K(\pi)$ the sectional curvature of $M$ associated with a plane section $\pi \subset T_pM$,

$p \in M$. For any orthonormal basis $e_1, \ldots, e_n$ of the tangent space $T_pM$, the scalar curvature $\tau$ at $p$ is defined to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j). \tag{2.8}$$

Let $L$ be a subspace of $T_pM$ of dimension $r \geq 2$ and $\{e_1, \ldots, e_r\}$ be an orthonormal basis of $L$. We define the scalar curvature $\tau(L)$ of the $r$-plane section $L$ by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r. \tag{2.9}$$

Given an orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space $T_pM$, we simply denote by $\tau_{1\ldots r}$ the scalar curvature of the $r$-plane section spanned by $e_1, \ldots, e_r$. The scalar curvature $\tau(p)$ of $M$ at $p$ is nothing but the scalar curvature of the tangent space of $M$ at $p$, and if $L$ is a 2-plane section, $\tau(L)$ is nothing but the sectional curvature $K(L)$ of $L$. Geometrically, $\tau(L)$ is nothing but the scalar curvature of the image $\exp_p(L)$ of $L$ at $p$ under the exponential map at $p$. For an integer $k \geq 0$ denote by $\mathcal{S}(n, k)$ the finite set consisting of unordered $k$-tuples $(n_1, \ldots, n_k)$ of integers $\geq 2$ satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. Denote by $\mathcal{S}(n)$ the set of unordered $k$-tuples with $k \geq 0$ for a fixed $n$. For each $k$-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ the two sequences of Riemannian invariants $\mathcal{S}(n_1, \ldots, n_k)(p)$ and $\hat{\mathcal{S}}(n_1, \ldots, n_k)(p)$ are defined respectively by

$$\begin{aligned} \mathcal{S}(n_1, \ldots, n_k)(p) &= \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \\ \hat{\mathcal{S}}(n_1, \ldots, n_k)(p) &= \sup\{\tau(L_1) + \cdots + \tau(L_k)\}, \end{aligned} \tag{2.10}$$

where $L_1, \ldots, L_k$ run over all $k$ mutually orthogonal subspaces of $T_pM$ such that $\dim L_j = n_j$, $j = 1, \ldots, k$. The two strings of Riemannian curvature invariants $\delta(n_1, \ldots, n_k)(p)$ and $\hat{\delta}(n_1, \ldots, n_k)(p)$ introduced by B. Y. Chen in [3] are given by

$$\delta(n_1, \dots, n_k)(p) = \tau(p) - \mathcal{S}(n_1, \dots, n_k)(p),$$

$$\hat{\delta}(n_1, \dots, n_k)(p) = \tau(p) - \hat{\mathcal{S}}(n_1, \dots, n_k)(p). \tag{2.11}$$

In terms of these $\delta$-invariants, the scalar curvature $\tau$ is nothing but $\delta(\varnothing) = \hat{\delta}(\varnothing)$ (with $k = 0$); moreover, the invariant $\delta_M$ introduced in [2] is nothing but the invariant $\delta(2)$ (with $k = 1$, $n_1 = 2$). Obviously, one has $\delta(n_1, \dots, n_k) \geq \hat{\delta}(n_1, \dots, n_k)$ for any $k$-tuple $(n_1, \dots, n_k)$ in $\mathcal{S}(n)$.

## 3. Sharp Inequalities Involving $\delta$-invariants

Let $M$ be an $n$-dimensional submanifold isometrically immersed in an $m$-dimensional locally conformal Kähler manifold $\widetilde{M}$. A locally conformal Kähler manifold $\widetilde{M}$ is said to be a *locally conformal Kähler space form* if the holomorphic sectional curvature is a real constant $c$ along $\widetilde{M}$. A locally conformal Kähler space form (resp. generalized Hopf space form) will be denoted by $\widetilde{M}(c)$ (resp. $\widetilde{M}(c, \tilde{\rho})$). Then, the Riemannian curvature tensor $\widetilde{R}$ on $\widetilde{M}(c)$ is given by

$$\begin{aligned}\widetilde{R}(X, Y)Z = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y + w(Y, Z)JX \\ &- w(X, Z)JY - 2w(X, Y)JZ\} \\ + \frac{3}{4}\{&g(Y, Z)P_1X - g(X, Z)P_1Y \\ &+ P(Y, Z)X - P(X, Z)Y\} \\ - \frac{1}{4}\{&w(Y, Z)\widetilde{P}_1X - w(X, Z)\widetilde{P}_1Y + \widetilde{P}(Y, Z)JX \\ &- \widetilde{P}(X, Z)JY - 2\widetilde{P}(X, Y)JZ - 2w(X, Y)\widetilde{P}_1Z\},\end{aligned} \tag{3.1}$$

where $g(P_1X, Y) = P(X, Y)$, $g(\widetilde{P}_1X, Y) = \widetilde{P}(X, Y)$.

A submanifold $M$ isometrically immersed in $\widetilde{M}(c)$ is called *totally real* if the almost complex structure $J$ of $\widetilde{M}(c)$ carries each tangent space

of $M$ into its corresponding normal space. On the other hand, for a totally real submanifold $M$ on $\widetilde{M}(c)$ we have $w(X, Y) = 0$ for vector fields $X, Y$ tangent to $M$. We denote by $R$ the Riemannian curvature tensor of $M$. Then, the Gauss equation on a totally real submanifold $M$ is given by

$$\begin{aligned} g(R(X, Y)Z, W) = {} & \frac{c}{4}\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} \\ & + \frac{3}{4}\{g(X, W)P(Y, Z) - g(Y, W)P(X, Z) \\ & + P(X, W)g(Y, Z) - P(Y, W)g(X, Z)\} \\ & + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)) \end{aligned} \tag{3.2}$$

for any vector fields $X, Y, Z, W$ tangent to $M$.

Furthermore, the scalar curvature $\tau$ of $M$ at $p$ is obtained by

$$2\tau(p) = n^2\| H \|^2 - \| h \|^2 + \frac{1}{4}n(n-1)(c + 6\sigma), \tag{3.3}$$

where $\| H \|^2$ and $\| h \|^2$ are the squared mean curvature and the squared norm of the second fundamental form, and we put

$$\sigma = \frac{1}{n}\sum_{i=1}^{n} P(e_i, e_i).$$

We give the following lemma for later use.

**Lemma 3.1** [2]. *Let* $a_1, \ldots, a_n, \eta$ *be* $n+1$ $(n \geq 2)$ *real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + \eta\right).$$

*Then,* $2a_1a_2 \geq \eta$, *with the equality holding if and only if* $a_1 + a_2 = a_3 = \cdots = a_n$.

For each $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, let $c(n_1, \ldots, n_k)$ and $b(n_1, \ldots, n_k)$ denote the positive constants given by

$$c(n_1, \dots, n_k) = \frac{n^2\left(n + k - 1 - \sum n_j\right)}{2\left(n + k - \sum n_j\right)}, \tag{3.4}$$

$$b(n_1, \dots, n_k) = \frac{1}{2}\left(n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)\right). \tag{3.5}$$

**Theorem 3.2.** *Let $M$ be an $n$-dimensional totally real submanifold of an $m$-dimensional locally conformal Kähler space form $\widetilde{M}(c)$ of constant holomorphic sectional curvature $c$. Then we have*

$$\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k)\| H \|^2 + b(n_1, \dots, n_k)\frac{c + 6\sigma}{4} \tag{3.6}$$

*for any $k$-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$.*

*The equality case of inequality (3.6) holds at a point $p \in M$ if and only if there exists an orthonormal basis $e_1, \dots, e_{2m}$ at $p$ such that the shape operators of $M$ in $\widetilde{M}(c)$ at $p$ take the following forms:*

$$A_r = \begin{pmatrix} A_1^r & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & A_k^r & \\ & 0 & & \mu_r I \end{pmatrix}, \quad r = n+1, \dots, 2m, \tag{3.7}$$

*where $I$ is an identity matrix and each $A_j^r$ are symmetric $n_j \times n_j$ submatrices such that*

$$trace(A_1^r) = \cdots = trace(A_k^r) = \mu_r. \tag{3.8}$$

**Proof.** Let $M$ be a totally real submanifold of a locally conformal Kähler space form $\widetilde{M}(c)$ of constant holomorphic sectional curvature $c$.

If $k = 1$, this was done in [7]. Hence, we assume $k > 1$.

Let $(n_1, \ldots, n_k) \in \mathcal{S}(n)$. Put

$$\eta = 2\tau - \frac{1}{4}n(n-1)(c+6\sigma) - \frac{n^2\left(n+k-1-\sum n_j\right)}{\left(n+k-\sum n_j\right)}\| H \|^2. \tag{3.9}$$

Substituting (3.3) in (3.9), we have

$$n^2\| H \|^2 = \gamma(\eta + \| h \|^2), \quad \gamma = n + k - \sum n_j. \tag{3.10}$$

Let $L_1, \ldots, L_k$ be mutually orthogonal subspaces of $T_pM$ with $\dim L_j = n_j$, $j = 1, \ldots, k$. By choosing an orthonormal basis $e_1, \ldots, e_{2m}$ at $p$ such that

$$L_j = \mathrm{Span}\{e_{n_1+\cdots+n_{j-1}+1}, \ldots, e_{n_1+\cdots+n_j}\}, \quad j = 1, \ldots, k \tag{3.11}$$

and $e_{n+1}$ is in the direction of the mean curvature vector, we obtain from (3.10) that

$$\left(\sum_{i=1}^{n} a_i\right)^2 = \gamma\left(\eta + \sum_{i=1}^{n}(a_i)^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,\, j=1}^{n}(h_{ij}^{r})^2\right), \tag{3.12}$$

where $a_i = h_{ii}^{n+1}$, $i = 1, \ldots, n$, and $\gamma = n + k - \sum n_j$.

We set

$$\Delta_1 = \{1, \ldots, n_1\}, \ldots, \Delta_k = \{n_1 + \cdots + n_{k-1} + 1, \ldots, n_1 + \cdots + n_k\}.$$

In other words, the equation (3.12) can be rewritten in the form

$$\left(\sum_{i=1}^{\gamma+1} \overline{a}_i\right)^2 = \gamma\left(\eta + \sum_{i=1}^{\gamma+1}(\overline{a}_i)^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,\, j=1}^{n}(h_{ij}^{r})^2 \right.$$
$$\left. - \sum_{2\leq\alpha_1\neq\beta_1\leq n_1} a_{\alpha_1}a_{\beta_1} - \sum_{\alpha_2\neq\beta_2} a_{\alpha_2}a_{\beta_2} - \cdots - \sum_{\alpha_k\neq\beta_k} a_{\alpha_k}a_{\beta_k}\right), \tag{3.13}$$

$\alpha_2, \beta_2 \in \Delta_2, \ldots, \alpha_k, \beta_k \in \Delta_k$

where we put

$$\overline{a}_1 = a_1,\ \overline{a}_2 = a_2 + \cdots + a_{n_1},$$

$$\overline{a}_3 = a_{n_1+1} + \cdots + a_{n_1+n_2},\ \ldots,\ \overline{a}_{k+1} = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+\cdots+n_k},$$

$$\overline{a}_{k+2} = a_{n_1+\cdots+n_k+1},\ \ldots,\ \overline{a}_{\gamma+1} = a_n. \tag{3.14}$$

Applying Lemma 3.1 to (3.13), we can obtain the following inequality

$$\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} + \sum_{\alpha_2<\beta_2} a_{\alpha_2}a_{\beta_2} + \cdots + \sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k}$$

$$\geq \frac{\eta}{2} + \sum_{i<j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m}\sum_{i,\ j=1}^{n}(h_{ij}^r)^2,$$

$$\alpha_j,\ \beta_j \in \Delta_j, \quad j = 1,\ \ldots,\ k. \tag{3.15}$$

Furthermore, from (2.9) and Gauss' equation we see that

$$\tau(L_j) = \frac{n_j(n_j-1)}{8}c + \frac{3}{4}n_j(n_j-1)\sigma$$

$$+ \sum_{r=n+1}^{2m}\sum_{\alpha_j<\beta_j}(h_{\alpha_j\alpha_j}^r h_{\beta_j\beta_j}^r - (h_{\alpha_j\beta_j}^r)^2),$$

$$\alpha_j,\ \beta_j \in \Delta_j, \quad j = 1,\ \ldots,\ k. \tag{3.16}$$

Thus, combining (3.15) and (3.16) we get

$$\tau(L_1) + \cdots + \tau(L_k)$$

$$\geq \frac{\eta}{2} + \sum_{j=1}^{k}\left(\frac{n_j(n_j-1)}{8}c + \frac{3}{4}n_j(n_j-1)\sigma\right)$$

$$+ \frac{1}{2}\sum_{r=n+1}^{2m}\sum_{(\alpha,\ \beta)\notin\Delta^2}(h_{\alpha\beta}^2)^2 + \frac{1}{2}\sum_{r=n+2}^{2m}\sum_{j=1}^{k}\left(\sum_{\alpha_j\in\Delta_j}h_{\alpha_j\alpha_j}^r\right)^2$$

$$\geq \frac{\eta}{2} + \sum_{j=1}^{k}\left(\frac{n_j(n_j-1)}{8}c + \frac{3}{4}n_j(n_j-1)\sigma\right), \tag{3.17}$$

where $\Delta = \Delta_1 \cup \cdots \cup \Delta_k$, $\Delta^2 = (\Delta_1 \times \Delta_1) \cup \cdots \cup (\Delta_k \times \Delta_k)$. Consequently, from (2.11), (3.9) and (3.17) we can obtain (3.6). If the equality in (3.6) holds at a point $p$, then the inequalities in (3.15) and (3.17) are actually equalities at $p$. In this case, by applying Lemma 3.1, (3.13), (3.15), (3.16) and (3.17), we also obtain (3.7) and (3.8). The converse can be verified by a straightforward computation.

**Corollary 3.3.** *Let $M$ be an $n$-dimensional Riemannian manifold and $p \in M$. If there exists a $k$-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ and a point $p \in M$ such that*

$$\delta(n_1, \ldots, n_k) > \frac{1}{8}\left(n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)\right)(c + 6\sigma),$$

*then $M$ admits no minimal totally real submanifold into any $m$-dimensional locally conformal Kähler space form $\overline{M}(c)$.*

As a consequence of Corollary 3.3, we obtain an obstruction to minimal immersions in a generalized Hopf space form.

**Corollary 3.4.** *Let $M$ be an $n$-dimensional Riemannian manifold and $p \in M$. If there exists a $k$-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ and a point $p \in M$ such that*

$$\delta(n_1, \ldots, n_k) > \frac{1}{8}\left(n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)\right)k, \quad k \in \mathbb{R},$$

*then $M$ admits no minimal totally real submanifold into any $m$-dimensional generalized Hopf space form $\widetilde{M}\left(k - \frac{6}{n}\widetilde{\rho}, \widetilde{\rho}\right)$, $\widetilde{\rho} \in \mathbb{R}$.*

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# CLASSIFICATION OF LOCALLY CONNECTED TORONTO SPACES

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## Abstract

In this article we shall answer the questions of J. Stepran's [Open Problems in Topology, Elsevier Science Publishers, B. V., Netherlands, 1990], about Toronto spaces under some additional condition by showing that every Hausdorff locally connected Toronto space of size $> \aleph_0$ is discrete.

## 1. Introduction

A Toronto space is a space $X$ which is homeomorphic to each of its subspaces of the same cardinality as $X$. So every discrete space is Toronto and locally connected.

Toronto problem asks whether it is possible to have an uncountable non-discrete Hausdorff space which is Toronto. Clearly the requirement that the space be Hausdorff and uncountable is natural, since cofinite topology on $\omega_1$ is a counterexample, and it can be shown that any Toronto space contains a countable discrete subspace [1].

Not much is known about the Toronto problem, except a few facts. First, any Hausdorff Toronto space is scattered and the number of

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isolated points in any non-discrete Hausdorff Toronto space is countable.

The author in [1] showed that if $(X, \tau)$ is a Hausdorff Toronto space of size $\aleph_1$, then each one of the following implies that $(X, \tau)$ is a discrete space.

(i) Countable union of closed subsets is closed,

(ii) $|bd(u)| \leq |u|$ for all open subsets $u$,

(iii) every open set is regularly open.

Further, he shows that every Hausdorff Toronto space of size $\geq \aleph_0$ is not compact.

There is a version of the problem which remains open and which might have some significance for the original question. For any ordinal $\alpha$, define an $\alpha$-*Toronto space* to be a scattered space of derived length $\alpha$, which is homeomorphic to each subspace of derived length $\alpha$.

**Question 1.1.** Is there an $\omega$-Toronto space?

**Question 1.2.** Is there an idempotent homogeneous filter on $\omega$?

The question concerning Toronto space of larger cardinalities and with stronger separation axioms also remain open.

**Question 1.3.** Is there some non-discrete Hausdorff Toronto space?

**Question 1.4.** Are all regular (or normal) Toronto spaces of size $\aleph_1$ discrete?

So Toronto problem is still open if $|X| \geq \aleph_1$ and $(X, \tau)$ is Hausdorff or regular or normal.

In this paper, we shall answer all of these questions about Toronto spaces, and show that every Hausdorff Toronto space of size $> \aleph_1$ is a discrete space.

## 2. Elementary Concepts

By a topological space $(X, \tau)$ we mean a set $X$ and a topology $\tau$ on it.

The topological closure of a set $A$ in $X$ is denoted by $\overline{A}$, the boundary of $A$ by $bd(A)$, exterior of $A$ by $\operatorname{Ext}(A) = X \setminus \overline{A}$ and interior of $A$ by $A^0$. A set $A$ in $X$ is called *regularly open* if $A = (\overline{A})^0$. The cardinality of $A$ is denoted by $|A|$, $A$ is denumerable if $|A| = |\mathbf{N}| = \aleph_0$, and $k = 2^{\aleph_{k-1}}$, and we assume that $m + n = \max\{m, n\}$, for all cardinals $m, n$. By $\omega$, we mean the space of ordinals $\{1, 2, ..., \omega_1\}$, $\omega_1$ is the first uncountable cardinal. A space $X$ is called *locally connected* if each $x \in X$ has a local base consisting of open connected subsets.

If $\mathcal{F}$ is a filter on $X$, then $\mathcal{F}^2$ is the filter on $X \times X$ defined by $A \in \mathcal{F}^2$ if and only if $\{a \in X : (b \in X : (a, b) \in A)\} \in \mathcal{F}$, and $\mathcal{F} \setminus A$ is the filter on $X \setminus A$ defined by $B \in \mathcal{F} \setminus A$ if and only if $B \cup A \in \mathcal{F}$. A filter $\mathcal{F}$ on $\omega$ is idempotent if $\mathcal{F}$ is isomorphic to $\mathcal{F}^2$ and it is homogeneous if $\mathcal{F}$ is isomorphic to $\mathcal{F} \setminus A$ for each $A \in \mathcal{F}$.

## 3. Main Result

**Theorem 3.1.** *Let $(X, \tau)$ be any Toronto Hausdorff topological space. Then $A = \{x : \{x\} \in \tau\}$ is dense.*

**Proof.** Suppose $A$ is not dense, then clearly $A$ is open and $\operatorname{Ext}(A) \neq \varnothing$. Since $\operatorname{Ext}(A)$, $A$, $bd(A)$ form a partition of $X$, one of these subsets must have the same cardinality as $X$, so we have the following three cases:

(1) $|\operatorname{Ext}(A)| = |X|$, since $(X, \tau)$ is Toronto, the subspace $\operatorname{Ext}(A)$ is homeomorphic to $(X, \tau)$, hence $\exists\, x_1 \in X$, such that $\{x_1\}$ is open in the subspace $\operatorname{Ext}(A)$. This implies that $\{x_1\} \in \tau$, which is impossible.

(2) $|bd(A)| = |X|$. Let $B = bd(A) \cup (A)$ so the subspace $B$ is homeomorphic to the space $(X, \tau)$.

Let $h : X \to B$ be a homeomorphism. Then $h(\operatorname{Ext}(A))$ is open in the

subspace $B$, but $h(\mathrm{Ext}(A)) \cap A = \varnothing$ so $h(\mathrm{Ext}(A)) \subseteq bd(A)$. Let $V \in \tau$ be such that $V \cap B = h(\mathrm{Ext}(A)) \neq \varnothing$, so $V \cap bd(A) \neq \varnothing$, which implies $V \cap B \neq \varnothing$. Let $t \in V \cap A$. Then $t \in h(\mathrm{Ext}(A)) \cap A$, which is impossible.

So the only remaining case is

(3) $|A| = |X|$, and so the subspace $A$ is homeomorphic to the space $X$, but clearly the subspace $A$ is a discrete space, so $X$ is discrete and hence $A = X$, which is impossible.

**Theorem 3.2.** *Let $(X, \tau)$ be any locally connected Hausdorff space and $|X| \geq \aleph_0$. If $(X, \tau)$ is Toronto, then it is a discrete space.*

**Proof.** Let $A = \{x : \{x\} \in \tau\}$, and suppose $|bd(A)| = |X|$, then the subspace $bd(A)$ is homeomorphic to the space $X$. So, there exists a point $x \in bd(A)$ such that $\{x\}$ is open. Then there exists $V \in \tau$ such that $V \cap bd(A) = \{x\}$, and hence the component $C(x)$ of $x$ in $V$ is open. But $V \subseteq A \cup \{x\}$, so $C(x) = \{x\}$, which implies $x \in A \cap bd(A) = \varnothing$.

**Remark.** In the previous proof we do not assume the continuum hypothesis because we proved the theorem for a topological space $(X, \tau)$, $|X| > \aleph_0$, which is stronger than the condition $|X| > \aleph_1$.

We complete the paper with the following question:

Is the following space a Toronto space?

**Example 3.3** (CH). Let $\{A_\alpha : \alpha \in [0, \omega_1]\}$ be a partition of $\mathbf{R}$ such that $A_0 = Q$, the set of rationals and $|A_\alpha| = \aleph_0$, $\forall \alpha \in [0, \omega_1]$. Let $B_\alpha = \bigcup_{\gamma < \alpha} A_\gamma$. Now we shall define a topology on $\mathbf{R}$ by defining a local base at each $x \in \mathbf{R}$. Let $x \in \mathbf{R}$, then $x \in A_\alpha$ for some $\alpha \in [0, \omega_1]$, and let $B_x = \{(a, b) \cap B_\alpha \cup \{x\} : (a, b)$ open interval contains $x\}$. Clearly, the topology is Hausdorff, $Q = \{x : \{x\} \in \tau\}$ and it is dense.

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# ON DIVIDED COMMUTATIVE SEMIGROUPS

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## Abstract

Let $S$ be a torsion-free, cancellative additive semigroup with identity 0 having total quotient group $G$. A prime ideal $P$ of $S$ is called divided if $P$ is comparable to every principal ideal of $S$. If every prime ideal of $S$ is divided, then $S$ is called a divided semigroup. If $P$ is a nonprincipal divided prime, then $P^{-1} = \{x \in G \mid x + P \subset P\}$ is a semigroup. We show that if $S$ is an atomic semigroup and divided, then the Krull dimension of $S \leq 1$. Also, we show that if a finitely generated prime ideal of a semigroup $S$ is divided, then it is maximal.

We study some semigroup versions of [1].

Throughout this paper, a semigroup will stand for a commutative cancellation torsion-free additive semigroup, and it is a (nonzero) semigroup with 0.

Let $S$ be a semigroup. If we set $G = \{s - s' \mid s, s' \in S\}$, then $G$ is a torsion-free abelian group with respect to addition and $S$ is a subsemigroup of $G$: $G$ is called the *quotient group* of $S$ and is denoted by $q(S)$. Any semigroup $T$ between $S$ and $q(S)$ is called an *oversemigroup* of $S$.

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*From now on, assume that $S$ is a semigroup with quotient group* $G = q(S)$.

We recall some notation and definitions ([2], [3], [4], [5] and [6]).

Let $I$ be a nonempty subset of $S$. Then $I$ is called an *ideal* of $S$ if $I + S \subset I$, that is, $a + s \in I$ for each $a \in I$ and each $s \in S$. An ideal $I$ is called a *proper ideal* if $I \neq S$. An element $u$ of $S$ is called a *unit* if $u + v = 0$ for some $v \in S$. Let $U(S)$ be the set of units in $S$. Note that $U(S)$ is a subgroup of $G$. If $M = S - U(S)$ is a nonempty subset of $S$, then $M$ is an ideal of $S$. Moreover, if $I$ is an ideal of $S$ such that $M \subset I$, then $M = I$ or $I = S$. Such a nonempty $M$ is called a *maximal ideal* of $S$. A proper ideal $P$ of $S$ is called a *prime ideal* of $S$ if $a + b \in P$ with $a, b \in S$ implies either $a \in P$ or $b \in P$. Let $\mathrm{Spec}(S)$ be the set of prime ideals of $S$. A proper ideal $Q$ of $S$ is called a *primary ideal* of $S$ if $a + b \in Q$ with $a, b \in S$ implies either $a \in Q$ or $nb \in Q$ for some positive integer $n$. For an ideal $I$ of $S$, put $\mathrm{rad}(I) = \{x \in S \mid nx \in I \text{ for some } n\}$. We say that $I$ is a *radical ideal* if $I = \mathrm{rad}(I)$. If $Q$ is a primary ideal of $S$, then $\mathrm{rad}(Q)$ is a prime ideal. Then we say that $Q$ is *P-primary* where $P = \mathrm{rad}(Q)$. If $S$ is a semigroup, a finite chain $P_1 \subset P_2 \subset \cdots \subset P_n$ of $n$ proper prime ideals of $S$ will be said to have *length* $n$. We define the (Krull) dimension of $S$ in terms of this concept. Suppose that there is a nonnegative integer $n$ such that $S$ contains a chain of prime ideals of length $n$, but no such chain of length $n + 1$. In this case, we say that $S$ has *dimension* $n$ or $S$ is *n-dimensional*, and we write $\dim S = n$.

A nonempty subset $N$ of a semigroup $S$ is called an *additive system* of $S$ if $a, b \in N$ implies $a + b \in N$ and $0 \in N$. Put $S_N = \{s - t \mid s \in S, t \in N\}$. Then $S_N$ is an oversemigroup of $S$ and is called the *quotient semigroup* of $S$ or *localization* of $S$. Each element $t \in N$ is a unit in $S_N$. If $P$ is a prime ideal of $S$, then $T = S - P$ is an additive system of $S$ and the quotient semigroup $S_T$ is denoted by $S_P$. Let $P \in \mathrm{Spec}(S)$. Then $S_P = \{s - t \mid s, t \in S, t \notin P\}$.

For any $x \in S$, put $(x) = x + S = \{x + a \mid a \in S\}$. Then $(x)$ is an ideal of $S$ and it is called a *principal ideal* of $S$. For $a_1, a_2, \ldots, a_n \in S$, we set $I = (a_1, a_2, \ldots, a_n) = \bigcup_{i=1}^{n}(a_i) = \bigcup_{i=1}^{n}(a_i + S)$. The ideal $(a_1, a_2, \ldots, a_n)$ of $S$ is called the *ideal generated by* $a_1, \ldots, a_n$, and $\{a_1, a_2, \ldots, a_n\}$ is called a *basis* of $I$. We say that $(a_1, a_2, \ldots, a_n)$ is a *finitely generated ideal* of $S$. Note that $(a + b) = (a) + (b)$ for each $a, b \in S$.

A nonempty subset $F$ of $q(S)$ is said to be a *fractional ideal* of $S$ if $x + F \subset S$ for some $x \in S$ and $F + S \subset F$. Each ideal of $S$ is a fractional ideal of $S$ and is called an *integral ideal*. For each $x \in q(S)$, $(x) = x + S$ is a fractional ideal.

In general, we set $(I_1 : I_2)_I = \{x \in I \mid x + I_2 \subset I_1\}$ for all subsets $I_1, I_2$ and $I$ of $q(S)$. If $I_1, I_2$ are fractional ideals of $S$ such that $I_2 \neq \emptyset$, then $(I_1 : I_2)_{q(S)}$ is a fractional ideal of $S$ and $(I_1 : I_2)_S$ is an integral ideal of $S$. A fractional ideal $F$ of $S$ is said to be *invertible* if there exists a fractional ideal $F'$ of $S$ such that $F + F' = S$. A principal fractional ideal is a invertible ideal.

Let $T$ be an oversemigroup of $S$. An element $t \in T$ is said to be *integral* over $S$ if $nt \in S$ for some positive integer $n$. The set $\overline{S}$ of elements $t \in T$ that are integral over $S$ is called the *integral closure* of $S$ in $T$. An oversemigroup $T$ is said to be *integral* over $S$ if each element $t \in T$ is integral over $S$. We say that $\overline{S}$ is the *integral closure* of $S$ if $\overline{S} = \{\alpha \in q(S) \mid \alpha \text{ is integral over } S\}$. Then $\overline{S}$ is a semigroup. That $S$ is said to be *integrally closed* if $\overline{S} = S$.

We say that $S$ is a *valuation semigroup* if $\alpha \in S$ or $-\alpha \in S$ for each $\alpha \in q(S)$. Also, $S$ is said to be *seminormal* if $2\alpha \in S$ and $3\alpha \in S$ for $\alpha \in q(S)$, then $\alpha \in S$. A valuation semigroup is a seminormal semigroup.

Put $\mathbf{Z_n} = \{a \in \mathbf{Z} \mid a \geq n\}$ where $\mathbf{Z}$ denotes the set of all integers.

We start with the following definitions.

**Definition 1.** A prime ideal $P$ of a semigroup $S$ is called *divided* if $P$ is comparable to every principal ideal of $S$. If every prime ideal of $S$ is divided, then $S$ is called a *divided semigroup*.

**Definition 2** (cf. [7]). A prime ideal $P$ of $S$ is called *strongly prime* if $a + P$ and $b + S$ are comparable for every $a, b \in S$. This condition is equivalent to the condition that if $\alpha + \beta \in P$ for $\alpha$ and $\beta \in q(S)$, then $\alpha \in P$ or $\beta \in P$. If every prime ideal of a semigroup $S$ is strongly prime, then $S$ is called a *pseudo-valuation semigroup* (in short, PVS).

**Example 1.** Put $S = \mathbf{Z}_0 \cup (\mathbf{Z} + (-\mathbf{Z}_1)X)$, where $X$ or $-X$ denotes an indeterminate, is a valuation semigroup and we have that $q(S) = \mathbf{Z} + \mathbf{Z}X$. Put $M = (1, -X) = (1 + S) \cup (-X + S)$ and $P = \mathbf{Z} + \mathbf{Z}_1(-X)$. Then $\operatorname{Spec}(S) = \{M, P\}$ and $M \underset{\neq}{\supset} P$.

Note that a valuation semigroup is a PVS, and a PVS is a seminormal semigroup.

**Proposition 1.** (1) *If $S$ is a divided semigroup, then the prime ideals of $S$ are linearly ordered.*

(2) *If $S$ is a PVS, then $S$ is a divided semigroup.*

**Proof.** (1) Let $P$ and $Q$ be prime ideals of $S$. Suppose that $P \not\subset Q$. Then there exists an element $a \in P - Q$. Hence $Q \subset (a) \subset P$. Therefore $Q \subset P$.

(2) Let $P$ be a prime ideal of $S$ and $a \in S$. Suppose that $P \not\supset (a)$. Then there exists an element $b \in (a) - P$. By definition, $P = 0 + P$ and $(a) = a + S$ are comparable. We have that $P \subset (a)$. Hence $P$ is divided.

**Proposition 2.** *The following statements are equivalent for a semigroup $S$.*

(1) *$S$ is a divided semigroup.*

(2) *For every pair of proper ideals $I$, $J$ of $S$, $I$ and $\operatorname{rad}(J)$ are comparable, where $\operatorname{rad}(J)$ denotes the radical of $J$, that is, $\operatorname{rad}(J) =$*

$\{x \in S \mid nx \in J$ *for some positive integer* $n\}$.

(3) *For every* $a, b \in S$, *the ideals* $(a)$ *and* $\mathrm{rad}((b))$ *are comparable.*

(4) *For every* $a, b \in S$, *either* $a \mid b$ *or* $b \mid na$ *for some* $n \geq 1$.

**Proof.** (1) $\Rightarrow$ (2). By Proposition 1(1), $\mathrm{rad}(J) = P$ is a prime ideal. Since $S$ is divided, we have that either $(a) \subset P$ or $(a) \supset P$ for each $a \in S$. Assume that $I \not\subset \mathrm{rad}(J)$. Then there exists an element $a \in I - P$. Then $P \subset (a) \subset I$. Hence, $\mathrm{rad}(J) \subset I$.

(2) $\Rightarrow$ (3). Clearly, (2) implies (3).

(3) $\Rightarrow$ (4). Assume that $a$ is not a divisor of $b$. If $\mathrm{rad}((b)) \subset (a)$, then $b \in (a)$. This is a contradiction. Therefore, we see that $(a) \subset \mathrm{rad}((b))$. Hence there exists a positive integer $n$ such that $na \in (b)$. Thus $b \mid na$.

(4) $\Rightarrow$ (1). Assume that $P \not\subset (a)$ for some $a \in S$ and $P \in \mathrm{Spec}(S)$. Then there exists an element $x \in P$ such that $x \notin (a) = a + S$. Therefore, $a$ is not a divisor of $x$. By (4), we have that $x \mid na$ for some positive integer $n$. Thus $na = x + s' \in P$ for some $s' \in S$. Hence $a \in P$, and so $(a) \subset P$.

**Example 2.** Let $S = \{(m, n) \in \mathbf{Z} \times \mathbf{Z} \mid m \geq 1, n \geq 1\} \cup (0, 0)$, where $\mathbf{Z}$ is the set of integers. Then $S$ is a semigroup with respect to addition. Also, $S$ is a divided semigroup but not a PVS. Then $\mathrm{Spec}(S) = \{S - \{0\}\}$.

**Proposition 3.** *Any localization of a divided semigroup is divided.*

**Proof.** Let $N$ be an additive system of a semigroup $S$ and $x, y \in S_N$. Then $x = a - s$ and $y = b - s$ for some $s \in N$ and $a, b \in S$. Since $S$ is divided, $b = a + c$ or $na = g + b$ for some $c, g \in S$ and $n \geq 1$. Thus, $b - s = c + (a - s)$ or $na - ns = (g - (n-1)s) + (b - s)$. Thus $x \mid y$ or $y \mid nx$. Therefore, $S_N$ is divided by Proposition 2(4).

**Proposition 4.** *The following statements are equivalent for a semigroup $S$ with the maximal ideal $M$.*

(1) *$S$ is divided.*

(2) *For every $a, b \in S$, there is an $n \geq 1$ such that $b + S$ and $na + M$ are comparable.*

**Proof.** (1) $\Rightarrow$ (2). Suppose that $b + S \not\subset ma + M$ for every $m \geq 1$. Then either $a$ and $b$ are associative in $S$ or $a$ does not divide $b$ in $S$. If $a$ and $b$ are associative, then $b \mid a$ and, therefore, $a + M \subset b + S$. If $a$ does not divide $b$ in $S$, then $b \mid na$ for some $n \geq 1$ by Proposition 2(4) and hence $na + M \subset b + S$.

(2) $\Rightarrow$ (1). Let $a, b \in S$. By Proposition 2(4), we need to show that either $a \mid b$ or $b \mid na$ for some $n \geq 1$. Now, if $b + S \subset na + M$ for some $n \geq 1$ or $a \notin M$, then $a \mid b$. If $ma + M \subset b + S$ for some $m \geq 1$ and $a \in M$, then $b \mid (m+1)a$. Indeed, if $a \in M$, then $ma + a \in ma + M \subset b + S$, and so $(m+1)a = b + s$ for some $s \in S$. Therefore, $b \mid (m+1)a$. Next, if $a$ is a unit of $S$, then $a \mid b$. Thus, $S$ is a divided semigroup.

Recall that if $I$ is an ideal of $S$, then $I^{-1} = \{x \in G \mid x + I \subset S\}$ and $I : I = \{x \in G \mid x + I \subset I\}$.

**Lemma 5.** *Let $I$ be a nonprincipal ideal of $S$ with the maximal ideal $M$. Then $x + I \subset M$ for every $x \in I^{-1}$.*

**Proof.** Put $J = x + I$. Since $J \subset S$, we have that $J$ is an ideal of $S$. Assume that $J \not\subset M$. Then $J = S$, and so $I = (-x) + S$. This contradicts the fact that $I$ is a nonprincipal ideal of $S$.

**Theorem 6.** *Let $P$ be a nonprincipal divided prime ideal of $S$. Then $P^{-1} = P : P$ and it is a semigroup.*

**Proof.** Suppose that there is an $x \in P^{-1} - S$. Write $x = a - b$ for some $a, b \in S$. Suppose that for some $p \in P$, $(a - b) + p = c \in S - P$. Then $a + p = b + c$ in $S$. Hence, $(a - b) + (p - c) = 0$ in $G$. Since $P$ is prime and $c \in S - P$, we deduce that $b \in P$. Since $P$ is divided and

$c \in S - P$, we obtain that $P \underset{\neq}{\subset} (c)$, and so $p - c \in P$. But $x + (p - c) = (a - b) + (p - c) = 0$, which is a contradiction by Lemma 5. Hence $x \in P : P$. Thus, $P^{-1} = P : P$. Also it is a semigroup.

**Proposition 7.** *Let $I$ be a proper ideal of $S$. Then the following statements are equivalent:*

*(1) $I$ is a nonprincipal divided prime ideal.*

*(2) $I^{-1}$ is a semigroup and $I$ is comparable to every principal ideal of $S$.*

**Proof.** (1) $\Rightarrow$ (2). This is clear by Theorem 6 and the definition of divided prime.

(2) $\Rightarrow$ (1). Since $I^{-1}$ is a semigroup, $I$ is nonprincipal. For, if $I$ were principal, then $I = (s)$ for some $s \in S$. Hence, $-s \in I^{-1}$. Since $I^{-1}$ is a semigroup, $-2s \in I^{-1}$. But $(-2s) + s = -s \notin S$, a contradiction. Now, we show that $I$ is prime. Let $N = S - I$ and $x, y \in N$. Since $I$ is comparable to every principal ideal of $S$, we obtain that $-x$ and $-y$ are in $I^{-1}$. Since $I^{-1}$ is a semigroup, $(-x) + (-y) = -(x + y) \in I^{-1}$. Since $I$ is nonprincipal and $-(x + y) \in I^{-1}$, we have that $x + y \in N$. Indeed, we have that $-(x + y) + I \subset S$, and so $I \underset{\neq}{\subset} (x + y)$. Suppose that $x + y \in I$. Then $(x + y) \subset I$, this is a contradiction. So $x + y \notin I$, that is, $x + y \in S - I = N$. Thus, $N$ is an additive system of $S$ and, therefore, $I$ is prime.

**Lemma 8.** (1) *The prime ideals of a semigroup $S$ are linearly ordered if and only if the radical of every proper ideal of $S$ is prime if and only if for every $a, b \in S$, either $a \mid nb$ or $b \mid ma$ for every $n, m \geq 1$.*

(2) *If $a, b \in S$, then $\operatorname{rad}((a)) = \operatorname{rad}((b))$ if and only if there are $n, m \geq 1$ such that $a \mid nb$ and $b \mid ma$.*

**Proof.** (1) Suppose that each proper radical ideal of $S$ is prime. Assume that $P, Q$ are two distinct prime ideals of $S$. Put $I = P \cap Q$.

Then $I = \mathrm{rad}(I)$. Hence $P \subset Q$ or $Q \subset P$. Therefore, $\mathrm{Spec}(S)$ is a linearly ordered set.

Next, we prove that, if for each $a, b \in S$, there are $m, n \geq 1$ such that $a \mid nb$ or $b \mid ma$ then $\mathrm{Spec}(S)$ is linearly ordered. Suppose that $P, Q$ are two distinct prime ideals of $S$. Assume that $P \not\subset Q$. Then there is a $p \in P - Q$. For every $q \in Q$, there is an $n \geq 1$ such that $p \mid nq$. Therefore, $q \in P$. Hence $Q \subset P$. Also, $\mathrm{Spec}(S)$ is linearly ordered if and only if the radical ideals of $S$ are linearly ordered if and only if each proper radical ideal of $S$ is prime if and only if the radical ideals of principal ideals of $S$ are linearly ordered.

(2) Just observe that $\mathrm{rad}((a)) = \mathrm{rad}((b))$ iff $a \in \mathrm{rad}((b))$ and $b \in \mathrm{rad}((a))$ iff there are $n, m \geq 1$ such that $a \mid nb$ and $b \mid ma$.

**Proposition 9.** *Suppose that the prime ideals of a semigroup $S$ with quotient group $G$ are linearly ordered, and $T$ is an oversemigroup of $S$ containing an element of the form $-s$ for some nonunit element $s \in S$. Furthermore, suppose that $\mathrm{rad}((s))$ is a minimal prime ideal of $S$. Then $T = G$. In particular, if $S$ is divided, then $T = G$ is divided.*

**Proof.** To show that $T = G$, it suffices to show that $-d \in T$ for every $d \in S$. We consider two cases:

**Case 1.** Suppose that $d \in S - \mathrm{rad}((s))$. Then $d \mid ns$ for some $n \geq 1$ by Lemma 8(1). Hence, $ns = d + k$ for some $k \in S$. Thus, $k - ns = -d$ in $G$. Since $-s \in T$, $k - ns = -d \in T$.

**Case 2.** Suppose that $d \in \mathrm{rad}((s))$. Since $\mathrm{rad}((s))$ is a minimal prime ideal of $S$ and $\mathrm{rad}((d))$ is prime by Lemma 8(1), $\mathrm{rad}((s)) = \mathrm{rad}((d))$. Hence, $d \mid ns$ for some $n \geq 1$ by Lemma 8(2). Now, a similar argument as in Case 1, we conclude that $-d \in T$. Thus, $T = G$. The remaining part is clear by Proposition 3.

**Corollary 10.** *Let $S$ be a semigroup of Krull dimension 1 containing a nonunit. Then $G$ is the only oversemigroup of $S$ containing an element of the form $-s$ for some nonunit $s \in S$.*

**Proposition 11.** *Let $P$ be a divided prime ideal of $S$. Then $nP$ is $P$-primary for every $n \geq 1$.*

**Proof.** We show that if $a, b \in S$ satisfy $a + b \in nP$ and $a \notin \text{rad}(nP) = P$, then $b \in nP$. Consider an element of the form $y = p_1 + p_2 + \cdots + p_n$ in $nP$, where the $p_i$'s are in $P$. To show $b \in nP$ it suffices to show that $y - a \in nP$, since $b$ is a finite sum of elements of the form of $y$. Since $a \notin P$ and $P$ is divided, we have that $P \underset{\neq}{\subset} (a)$, and so $p - a \in P$ for every $p \in P$. For, we have that $p - a \in S$ and so $a + s = p \in P$ for some $s \in S$. Since $a \notin P$, we have that $s \in P$, and so $p - a \in P$. Thus, $y - a = (p_1 - a) + p_2 + \cdots + p_n \in nP$. Hence, $b \in nP$. Therefore, $nP$ is $P$-primary.

Recall that $S$ is *atomic* if every nonunit of $S$ is a finite sum of irreducible elements.

**Theorem 12.** *Let $S$ be an atomic semigroup. Then $S$ is divided if and only if $S$ is of Krull dimension* 1.

**Proof.** Suppose that $S$ is divided with maximal ideal $M$. Suppose that there is a prime ideal $P$ of $S$ such that $P \underset{\neq}{\subset} M$. Then there are irreducible elements $a, b \in S$ such that $a \in P$ and $b \in M - P$. Since $P$ is divided, $b \mid a$ which is a contradiction. The converse is clear.

**Lemma 13.** *Let $s$ be a nonunit element of $S$. Then $-s$ is never integral over $S$.*

**Proof.** Suppose that $-s$ is integral over $S$. Then there is a positive integer $n$ such that $n(-s) \in S$. Therefore, $n(-s) = t$ where $t \in S$, and so $0 = t + ns$. Hence $s$ is a unit, a contradiction.

**Lemma 14** (cf. [2, Theorem 1]). *Let $S$ be a semigroup and let $I$ be a finitely generated ideal of $S$ such that $x \in I : I$. Then $x$ is integral over $S$.*

**Proof.** Let $I = (a_1, a_2, \ldots, a_m)$. Since $x \in I : I$, we have that $a_1 + x = a_{i_1} + s_1$ where $s_1 \in S$. If $i_1 = 1$, then $a_1 + x = a_1 + s_1$, so that $x = s_1 \in S$. If $i_1 \neq 1$, then $a_{i_1} + x = a_{i_2} + s_2$ for $s_2 \in S$. Moreover, if

$i_2 = 1$, then $a_{i_1} + x = a_1 + s_2$. Therefore, $(a_1 + x) + (a_{i_1} + x) = (a_{i_1} + s_1) + (a_1 + s_2)$, so that $2x = s_1 + s_2 \in S$. Repeating this process, $nx = s_1 + s_2 + \cdots + s_n \in S$, where $s_i \in S$ and $1 \leq n \leq k$. Consequently, we see that $x$ is integral over $S$.

**Theorem 15.** *Let $P$ be a finitely generated prime ideal of $S$. If $P$ is divided, then $P$ is maximal.*

**Proof.** Let $M$ be the maximal ideal of $S$. Suppose that $P$ is not maximal. Then there is an $s \in M - P$. Since $P$ is prime and divided, we have that $P \underset{\neq}{\subset} (s)$. Therefore, $-s + P \subset S$, and so $-s + p = t$ for some $p \in P$ and some $t \in S$. Since $p = s + t \in P$ and $s \notin P$, we have that $t \in P$. Hence $-s + P \subset P$. Since $P$ is a finitely generated ideal, $-s$ is integral over $S$ by Lemma 14. This is a contradiction to the fact that $-s$ is never integral over $S$ by Lemma 13. Hence $M = P$.

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# APPLICATIONS OF CONTINUED FRACTIONS

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## Abstract

The purpose of this article is to provide an overview of some striking applications of continued fractions to the problem of factoring integers and the implications for cryptography, as well as to finding solutions of Diophantine equations. In particular, we provide simple necessary and sufficient conditions solely in terms of simple continued fractions for all solutions of the Diophantine equation $x^2 - Dy^2 = n$.


## 1. Notation and Preliminaries

This section is devoted to developing the interrelationships among continued fractions, ideals and certain irrational numbers that we now define.

Suppose that $D \in \mathbb{N}$ is not a perfect square. Then a *quadratic irrational* is a number of the form

$$\alpha = \frac{P + \sqrt{D}}{Q}, \quad (P, Q \in \mathbb{Z})$$

where $Q \neq 0$ and $P^2 \equiv D \pmod{Q}$. Also, the *algebraic conjugate of* $\alpha$ is

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$$\alpha' = \frac{P - \sqrt{D}}{Q}.$$

Continued fraction expansions will be denoted by

$$\langle q_0; q_1, q_2, \ldots, q_j, \ldots \rangle,$$

where $q_j \in \mathbb{R}$ are called the *partial quotients* of the continued fraction expansion. If $q_j \in \mathbb{Z}$, and $a_j > 0$ for all $j > 0$, then the continued fraction is called an *infinite simple continued fraction* (which is equivalent to being an irrational number), whereas if the expression terminates, then it is called a *finite simple continued fraction* (which is equivalent to being a rational number). See [3, Theorem 5.1.1, p. 223; Theorems 5.2.1-5.2.2, pp. 228-229].

To establish the link with continued fractions, we first note that it is well-known that a real number has a periodic continued fraction expansion if and only if it is a quadratic irrational (see [3, Theorem 5.3.1, p. 240]). Furthermore, a quadratic irrational *may* have a *purely* periodic continued fraction expansion which we denote by

$$\alpha = \langle \overline{q_0; q_1, q_2, \ldots, q_{\ell-1}} \rangle$$

meaning that $q_n = q_{n+\ell}$ or all $n \geq 0$, where $\ell = \ell(\alpha)$ is the period length of the simple continued fraction expansion. It is known that a quadratic irrational $\alpha$ *has* such a purely periodic expansion if and only if $\alpha > 1$ and $-1 < \alpha' < 0$. Any quadratic irrational which satisfies these two conditions is called *reduced* (see [3, Theorem 5.3.2, p. 241]). Now we are in a position to bring in the theory of ideals and link the three together.

Let $D > 1$ be a square-free positive integer and set:

$$\sigma = \begin{cases} 2 & \text{if } D \equiv 1 \pmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Define

$$\Delta = 4D/\sigma^2,$$

and

$$\omega_\Delta = (\sigma - 1 + \sqrt{D})/\sigma,$$

then

$$\Delta = (\omega_\Delta - \omega'_\Delta)^2.$$

The value $\Delta$ is called a *fundamental discriminant*, *field discriminant* or simply *discriminant* with associated *radicand* $D$, and $\omega_\Delta$ is called the *principal fundamental surd associated with* $\Delta$. This will provide the canonical basis element for certain rings that we now define.

Let $[\alpha, \beta] = \alpha\mathbb{Z} + \beta\mathbb{Z}$ be a $\mathbb{Z}$-module. Then

$$\mathcal{O}_\Delta = [1, \omega_\Delta],$$

is called the *ring of integers in K*.

It may be shown that any $\mathbb{Z}$-module $I \neq (0)$ of $\mathcal{O}_\Delta$ has a representation of the form $[a, b + c\omega_\Delta]$, where $a, c \in \mathbb{N}$ with $0 \leq b < a$. We will only be concerned with *primitive* ones, namely those for which $c = 1$. In other words, $I$ is a primitive $\mathbb{Z}$-submodule of $\mathcal{O}_\Delta$ if whenever $I = (z)J$ for some $z \in \mathbb{Z}$ and some $\mathbb{Z}$-submodule $J$ of $\mathcal{O}_\Delta$, then $|z| = 1$. Thus, a canonical representation of a primitive $\mathbb{Z}$-submodule of $\mathcal{O}_\Delta$ is obtained by setting $\sigma a = Q$ and $b = (P-1)/2$ if $\sigma = 2$, while $b = P$ if $\sigma = 1$ for $P, Q \in \mathbb{Z}$, namely

$$I = [Q/\sigma, (P + \sqrt{D})/\sigma]. \tag{1.1}$$

Now we set the stage for linking ideal theory with continued fractions by giving a criterion for a primitive $\mathbb{Z}$-module to be a primitive ideal in $\mathcal{O}_\Delta$. A nonzero $\mathbb{Z}$-module $I$ as given in (1.1) is called a *primitive* $\mathcal{O}_\Delta$*-ideal* if and only if $P^2 \equiv D \pmod{Q}$ (see [3, Theorem 3.5.1, p. 173]). *Henceforth, when we refer to an* $\mathcal{O}_\Delta$*-ideal it will be understood that we mean a primitive* $\mathcal{O}_\Delta$*-ideal*. Also, the value $Q/\sigma$ is called the *norm of I*, denoted by $N(I)$. Hence, we see that $I$ is an $\mathcal{O}_\Delta$-ideal if and only if

$\alpha = (P + \sqrt{D})/Q$ is a quadratic irrational. Given the notion of a reduced quadratic irrational discussed earlier, it is not surprising that we define a *reduced ideal* $I$ to be one which contains an element $\beta = (P + \sqrt{D})/\sigma$ such that $I = [N(I), \beta]$, where $\beta > N(I)$ and $-N(I) < \beta' < 0$, since this corresponds exactly to the reduced quadratic irrational $\alpha = \beta/N(I) > 1$ with $-1 < \alpha' < 0$. In fact, the following holds.

**Theorem 1.1.** *Let $\Delta$ be a discriminant with associated radicand $D$. Then $I = [Q/\sigma, (P + \sqrt{D})/\sigma]$ is a reduced $\mathcal{O}_\Delta$-ideal if $Q/\sigma < \sqrt{\Delta}/2$. Conversely, if $I$ is reduced, then $Q/\sigma < \sqrt{\Delta}$. Furthermore, if $0 \le P - \sigma + 1 < Q < 4\sqrt{D}$ and $Q > \sqrt{D}$, then $I$ is reduced if and only if $Q - \sqrt{D} < P < \sqrt{D}$.*

**Proof.** See [2, Corollaries 1.4.2-1.4.4, p. 19].

Also, it is not surprising that we define the *conjugate ideal* of $I$ to be

$$I' = [Q/\sigma, (P - \sqrt{D})/\sigma].$$

Now we link continued fractions to the ideals defined above. Let $I$ be an $\mathcal{O}_\Delta$-ideal given by (1.1). Define

$$P_0 = P,\ Q_0 = Q, \text{ and recursively for } j \ge 0,$$

$$q_j = \left\lfloor \frac{P_j + \sqrt{D}}{Q_j} \right\rfloor, \tag{1.2}$$

$$P_{j+1} = q_j Q_j - P_j, \tag{1.3}$$

and

$$D = P_{j+1}^2 + Q_j Q_{j+1}. \tag{1.4}$$

It follows that we have the simple continued fraction expansion of the quadratic irrational:

$$\alpha = \frac{P + \sqrt{D}}{Q} = \frac{P_0 + \sqrt{D}}{Q_0} = \langle q_0; q_1, \ldots, q_j, \ldots \rangle.$$

Now the stage is set for the appearance of the result that formally merges ideals and continued fractions. We only need the notion of the equivalence of two $\mathcal{O}_\Delta$-ideals $I$ and $J$, denoted by $I \sim J$ to proceed. We write $I \sim J$ to denote the fact that there exist nonzero integers $\alpha, \beta \in \mathcal{O}_\Delta$ such that $(\alpha)I = (\beta)J$, where $(x)$ denotes the principal $\mathcal{O}_\Delta$-ideal generated by $x \in \mathcal{O}_\Delta$. For a given discriminant $\Delta$, the *class group* of $\mathcal{O}_\Delta$ determined by these equivalence classes, denoted by $\mathfrak{C}_\Delta$, is of finite order, denoted by $h_\Delta$, called the *class number* of $\mathcal{O}_\Delta$.

Now we may present the *Continued Fraction Algorithm.*

**Theorem 1.2.** *Suppose that $\Delta \in \mathbb{N}$ is a discriminant, $P_j$, $Q_j$ are given by* (1.2)-(1.4), *and*

$$I_j = [Q_{j-1}/\sigma, (P_{j-1} + \sqrt{D})/\sigma]$$

*for nonnegative $j \in \mathbb{Z}$. Then $I_1 \sim I_j$ for all $j \in \mathbb{N}$. Furthermore, there exists a least natural number $n$ such that $I_{n+j}$ is reduced for all $j \geq 0$, and these $I_{n+j}$ are all of the reduced ideals equivalent to $I_1$. If $\ell \in \mathbb{N}$ is the least value such that $I_n = I_{\ell+n}$, then for $j \geq n-1$, $\alpha_j = (P_j + \sqrt{D})/Q_j$ all have the same period length $\ell = \ell(\alpha_j) = \ell(\alpha_{n-1})$.*

**Proof.** See [3, Theorem 5.5.2, pp. 261-266].

**Remark 1.1.** From the continued fraction algorithm, we see that if

$$I = [Q/\sigma, (P + \sqrt{D})/\sigma]$$

is a reduced $\mathcal{O}_\Delta$-ideal, then the set

$$\{Q_1/\sigma, Q_2/\sigma, \ldots, Q_\ell/\sigma\}$$

represents the norms of all reduced ideals equivalent to $I$. This is achieved via the simple continued fraction expansion of $\alpha = (P + \sqrt{D})/Q$.

An important result needed in the next section is given as follows.

**Theorem 1.3.** *Suppose that* $\Delta$ *is a discriminant and* $\ell(\omega_\Delta) = \ell$ *with* $Q_j$ *defined for the simple continued fraction expansion of* $\omega_\Delta$ *as in Equations* (1.2)-(1.4). *Then* $Q_j/\sigma | D$ *with* $0 < j < \ell$ *if and only if* $j = \ell/2$.

**Proof.** See [2, Theorem 6.1.4, p. 193].

In the next section, we require results on the following well-known sequences. For a quadratic irrational $\alpha = \langle q_0; q_1, \ldots\rangle$, define two sequences of integers $\{A_j\}$ and $\{B_j\}$ inductively by:

$$A_{-2} = 0,\ A_{-1} = 1,\ A_j = q_j A_{j-1} + A_{j-2} \quad (\text{for } j \geq 0), \tag{1.5}$$

$$B_{-2} = 1,\ B_{-1} = 0,\ B_j = q_j B_{j-1} + B_{j-2} \quad (\text{for } j \geq 0). \tag{1.6}$$

By [3, Theorem 5.1.3, p. 225], we have the following identity for any $j \in \mathbb{N}$:

$$A_j B_{j-1} - A_{j-1} B_j = (-1)^{j-1}. \tag{1.7}$$

Also, if $\alpha = (P_0 + \sqrt{D})/Q_0$, and if we set

$$G_{j-1} = Q_0 A_{j-1} - P_0 B_{j-1} \quad (\text{for } j \geq 1),$$

then by [3, Theorem 5.3.4, p. 246],

$$G_{j-1}^2 - B_{j-1}^2 D = (-1)^j Q_j Q_0 \quad (\text{for } j \geq 1). \tag{1.8}$$

It is also known that the above sequences are related to continued fraction expansions in the following manner.

Let $n \in \mathbb{N}$ and let $\alpha$ have a continued fraction expansion

$$\langle q_0; q_1, \ldots, q_n, \ldots\rangle$$

with $q_j \in \mathbb{R}^+$ when $j > 0$. Then

$$C_k = \langle q_0; q_1, \ldots, q_k\rangle$$

is the $k^{\text{th}}$ convergent of $\alpha$ for any nonnegative integer $k \leq n$, and

$$C_k = A_k/B_k = \frac{q_k A_{k-1} + A_{k-2}}{q_k B_{k-1} + B_{k-2}}, \tag{1.9}$$

is the $k^{\text{th}}$ convergent of $\alpha$ for any nonnegative integer $k \leq n$.

## 2. Applications to Factoring

Given the fact that certain cryptosystems such as the RSA Public-key Cipher (see [6]) depend, for their security, upon the intractability of factoring large integers, it is valuable to have a look at some factorization algorithms, by which we mean an algorithm that solves the problem of determining the complete factorization of an integer $n > 1$ guaranteed by the Fundamental Theorem of Arithmetic. In other words, the algorithm should (*ultimately*, as described below) find distinct primes $p_j$ and $a_j \in \mathbb{N}$ such that $n = \prod_{j=1}^{k} p_j^{a_j}$. We observe that it suffices for such algorithms to merely find $r, s \in \mathbb{N}$ such that $1 < r \leq s < n$ with $n = rs$ (called *splitting* $n$), since we can then apply the algorithm to $n/r$ and to $s$, thereby recursively splitting each composite number until a complete factorization is found.

We will concentrate upon the factoring method related to continued fractions that reigned from its birth in mid 1970 until the Quadratic Sieve (see [6, pp. 203-206]) took over in the early 1980s. The history leading up to the inception of the continued fraction algorithm is a rich one.

In 1643, Fermat developed a method for factoring that was based upon a simple observation. If $n = rs$ is an odd natural number with $r < \sqrt{n}$, then

$$n = a^2 - b^2 \text{ where } a = (s+r)/2 \text{ and } b = (s-r)/2.$$

Hence, in order to find a factor of $n$, we need only look at values $x = a^2 - n$ for $a = \lfloor \sqrt{n} \rfloor + 1,\ \lfloor \sqrt{n} \rfloor + 2,\ \ldots,\ (n-1)/2$ until a perfect square is found. This is called the *difference of squares method* of factoring, and it

has been rediscovered numerous times. Legendre exploited the idea as follows. He looked at congruences $x^2 \equiv \pm py^2 \pmod{n}$ for primes $p$. A solution to this congruence means that $\pm p$ is a quadratic residue modulo all prime factors of $n$. Hence, if the residue is $+2$, for instance, then we know that all prime factors of $n$ are congruent to $\pm 1$ modulo 8, so already we have halved the search for factors of $n$. Legendre applied this method repeatedly for various primes $p$. Thus, what Legendre was essentially doing was to construct a quadratic sieve (by which we mean a sieve in which about half of the possible numbers being sieved are removed from consideration) by getting lots of residues modulo $n$, thereby eliminating potential prime divisors of $n$ that sit in various linear sequences. He found that if one could get enough of these, then one could eliminate primes up to $\sqrt{n}$ as prime divisors and thus show $n$ was prime. What underlies this method of Legendre is the continued fraction expansion of $\sqrt{n}$ since essentially what Legendre was doing was finding *small* residues modulo $n$. It should also be noted that, in retrospect, what Euler did could be used to get equation (2.10) below as follows. Euler considered two representations of $n$:

$$n = x^2 + ay^2 = z^2 + aw^2,$$

so

$$(xw)^2 \equiv (n - ay^2)w^2 \equiv nw^2 - ay^2w^2 \equiv -ay^2w^2$$

$$\equiv (z^2 - n)y^2 \equiv (zy)^2 \pmod{n},$$

and we are back to a potential factor for $n$. The basic idea in the above, for a given $n \in \mathbb{N}$, is simply that if we can find $x, y \in \mathbb{Z}$ such that

$$x^2 \equiv y^2 \pmod{n}, \tag{2.10}$$

and $x \not\equiv \pm y \pmod{n}$, then $\gcd(x - y, n)$ is a nontrivial factor of $n$. This idea is currently exploited by numerous algorithms. For instance, the continued fraction algorithm uses the construction of these $(x, y)$ pairs as follows.

First we need to know that a *factor base* is a "small" set of primes that will remain under consideration for the duration of the discussion of the algorithm. Moreover, given $B \in \mathbb{N}$, $n \in \mathbb{N}$ is said to be a *B-smooth number* or simply *B-smooth* if all primes dividing $n$ are no bigger than $B$. $B$ itself is called a *smoothness bound* (see [6] for more detail).

## The Continued Fraction Factoring Algorithm

Suppose that we wish to factor $n \in \mathbb{N}$ and a smoothness bound $B$ has been selected. Then execute the following steps:

(1) Choose a factor base of primes $\mathcal{F} = \{p_1, p_2, ..., p_k\}$ for some $k \in \mathbb{N}$ determined by $B$ and a large upper index value $J$. (From the knowledge about the distribution of smooth integers close to $\sqrt{n}$, the optimal $k$ is known to be one which is chosen to be approximately $\sqrt{\exp(\sqrt{\log(n)\log\log(n)})}$.)

(2) Let $\alpha = \sqrt{n}$ and use Equations (1.2)-(1.4) to define the values $Q_j$. Then trial divide $Q_j$ by the primes in $\mathcal{F}$ to determine if $Q_j$ is $p_k$-smooth. If it is, use its factorization $Q_j = \prod_{i=1}^{k} p_i^{a_{i,j}}$ to form the binary $k+1$-tuple $\mathfrak{b}_j = (v_{0,j}, v_{1,j}, v_{2,j}, \ldots, v_{k,j})$, where $v_{0,j}$ is respectively 0 or 1 according as $j$ is even or odd, and for $1 \le i \le k$, $v_{i,j}$ is respectively 0 or 1 according as $a_{i,j}$ is even or odd. If $Q_j$ is not $p_k$-smooth, discard it and return to calculate $Q_{j+1}$.

(3) For each set $\mathcal{S}$ of the vectors $\mathfrak{b}_j$ constructed in (2), for which it is discovered that

$$\sum_{j \in \mathcal{S}} v_{i,j} \equiv 0 \pmod 2, \quad 0 \le i \le k,$$

we have $x^2 \equiv y^2 \pmod n$, where

$$x = \left[\prod_{j \in \mathcal{S}} (-1)^j Q_j\right]^{1/2} \quad \text{and} \quad y \equiv \prod_{j \in \mathcal{S}} A_{j-1} \pmod n.$$

If $x \not\equiv \pm y \pmod{n}$, then $\gcd(x \pm y, n)$ gives a nontrivial factor of $n$. (The computational details for finding sets $\mathcal{S}$ are given in [7].)

By Equation (1.8), $A_{j-1}^2 - nB_{j-1}^2 = (-1)^j Q_j$, which is the heart of the algorithm. Thus, we have that $nB_{j-1}^2 \equiv A_{j-1}^2 \pmod{p}$, for any prime $p \mid Q_j$, so $n$ is a quadratic residue modulo $p$. Hence, we only put primes $p$ in the factor base for which $n$ is a quadratic residue modulo $p$. The following gives a small illustration of the continued fraction algorithm, called CFRAC, a term coined by Sam Wagstaff.

**Example 2.1.** Let $n = 6109$. Our factor base will be $\mathcal{F} = \{3, 5, 11, 13, 31, 37\}$. Since $\lfloor\sqrt{n}\rfloor = 78$, we compute the following table (where $J = 3$).

| $j$ | $P_j$ | $q_j$ | $A_{j-1}$ | $(-1)^j Q_j$ | $\mathfrak{v}_j$ |
|---|---|---|---|---|---|
| 0 | 0 | 78 | 1 | 1 | (0, 0, 0, 0, 0, 0, 0) |
| 1 | 78 | 6 | 78 | −25 | (1, 0, 0, 0, 0, 0, 0) |
| 2 | 72 | 4 | 469 | 37 | (0, 0, 0, 0, 0, 0, 1) |
| 3 | 76 | 17 | 1954 | −9 | (1, 0, 0, 0, 0, 0, 0) |

We have a set $\mathcal{S}$ such that $\sum_{j\in\mathcal{S}} v_{i,\,j} \equiv 0 \pmod{2}$ for each $i = 0, 1, \ldots, 6$. This set is $\mathcal{S} = \{1, 3\}$ for which we have $Q_1 = -5^2$, $Q_3 = -3^2$, $A_0 = 78$ and $A_2 = 1954$. We compute $\prod_{j\in\mathcal{S}} A_{j-1} \equiv 5796 \pmod{6109}$ and since

$$y^2 = \prod_{j\in\mathcal{S}} A_{j-1}^2 \equiv x^2 = \prod_{j\in\mathcal{S}} Q_j = 15^2 \pmod{n},$$

then we check $\gcd(x \pm y, n)$. We compute that both $\gcd(x - y, n) = \gcd(15 - 5796, 6109) = 41$, and $\gcd(x + y, n) = \gcd(15 + 5796, 6109) = 149$. Thus, we have factored $n = 41 \cdot 149$.

Although we have only looked at the continued fraction method for factoring, there are numerous others which have proven to be powerful in the modern day. The reader may consult [6] for a description of the *quadratic sieve, the number field sieve*, and the *elliptic curve* method for factoring, among others.

### 3. Solutions of Diophantine Equations

Perhaps one of the most striking applications of continued fractions is the criterion for solvability of Pell's equation:

$$x^2 - Dy^2 = \pm 1,$$

for integers $x$, $y$. The solution is given as follows.

**Theorem 3.1.** *Suppose that* $D \in \mathbb{N}$ *is not a perfect square and*

$$\alpha = \sqrt{D} = \langle q_0; \overline{q_1, q_2, \ldots, q_\ell} \rangle$$

*with* $\ell = \ell(\sqrt{D})$. *If* $\ell$ *is even, then all positive solutions of*

$$x^2 - y^2 D = 1 \tag{3.11}$$

*are given by*

$$x = A_{k\ell-1} \quad and \quad y = B_{k\ell-1}$$

*for* $k \in \mathbb{N}$, *whereas there are no solutions to*

$$x^2 - y^2 D = -1. \tag{3.12}$$

*If* $\ell$ *is odd, then all positive solutions of Equation* (3.11) *are given by*

$$x = A_{2k\ell-1} \quad and \quad y = B_{2k\ell-1}$$

*for* $k \in \mathbb{N}$, *whereas all positive solutions of Equation* (3.12) *are given by*

$$x = A_{(2k-1)\ell-1} \quad and \quad y = B_{(2k-1)\ell-1}$$

*for* $k \in \mathbb{N}$.

**Proof.** See [3, Corollary 5.3.3, p. 249].

**Example 3.1.** Let $D = 65$. Then $\sqrt{D} = \langle 8; \overline{16} \rangle$, so $\ell(\sqrt{65}) = 1$. We calculate that $A_0 = 8$, $A_1 = 129$, $A_2 = 2072$, $A_3 = 33281$, ..., and $B_0 = 1$, $B_1 = 16$, $B_2 = 257$, $B_3 = 4128$, ..., so by Theorem 3.1, all of the infinitely many solutions to (3.11) are given by

$$A_{2k-1}^2 - B_{2k-1}^2 \cdot 65 = 1,$$

for all $k \in \mathbb{N}$. In particular,

$$A_1^2 - B_1^2 \cdot 65 = 129^2 - 16^2 \cdot 65 = A_3^2 - B_3^2 \cdot 65$$

$$= 33281^2 - 4128^2 \cdot 65 = 1.$$

Also, all of the infinitely many solutions of (3.12) are given by

$$A_{2k-2}^2 - B_{2k-2}^2 \cdot 65 = -1,$$

for all $k \in \mathbb{N}$, so in particular,

$$A_0^2 - B_0^2 \cdot 65 = 8^2 - 1^2 \cdot 65 = A_2^2 - B_2^2 \cdot 65 = 2072^2 - 257^2 \cdot 65 = -1.$$

From Equation (1.9) and Theorem 3.1 we see that if $x^2 - y^2 D = \pm 1$, for nonsquare $D$, has a solution $x, y \in \mathbb{Z}$, then $x/y = A_j/B_j = C_j$ is a convergent in the simple continued fraction expansion of $\sqrt{D}$. There is a more general result.

**Theorem 3.2.** *Let $D \in \mathbb{N}$ be not a perfect square, and $n \in \mathbb{Z}$ with $|n| < \sqrt{D}$. If*

$$x^2 - Dy^2 = n, \tag{3.13}$$

*for some $x, y \in \mathbb{N}$, then there exists a nonnegative integer $j$ such that $x/y = A_j/B_j = C_j$ is a convergent in the simple continued fraction expansion of $\sqrt{D}$.*

**Proof.** See [3, Theorem 5.2.5, p. 232].

**Remark 3.1.** From Theorem 3.2 and Equation (1.8) we see that, in the case where $|n| < \sqrt{D}$, Equation (3.13) is solvable if and only if there exists a nonnegative integer $j$ such that, in the simple continued fraction expansion of $\sqrt{D}$, $n = (-1)^j Q_j$, $x = A_{j-1}$, and $y = B_{j-1}$. The case where $|n| > \sqrt{D}$ is solved in [3, 4] using ideal theory in conjunction with what are called *semi-simple continued fractions*. Below we extract information from [3] that handles this case and we incorporate the result in Theorem 3.2 as well using only simple continued fraction expansions.

**Theorem 3.3.** *Let $n \in \mathbb{Z}$ be given and fixed and set $Q_0 = n$. Then Equation* (3.13) *is solvable in relatively prime integers $x_0$, $y_0$ if and only if*

(a) *There exists a unique integer $P_0$ where $P_0^2 \equiv D \pmod{|Q_0|}$ for $-|Q_0|/2 < P_0 \le |Q_0|/2$ such that $Q_t = 1$ for some integer $t$ in the simple continued fraction expansion of $\alpha_0 = (P_0 + \sqrt{D})/Q_0$*

*and*

(b) *There exists a nonnegative integer $m$ such that $\ell m + t - 1$ is odd (where $\ell = \ell(\alpha_0)$), $x_0 = G_{\ell m+t-1}$, and $y_0 = B_{\ell m+t-1}$ in the simple continued fraction expansion of $\alpha_0$.*

**Proof.** If (a)-(b) hold, then by Equation (1.8),

$$G_{\ell m+t-1}^2 - B_{\ell m+t-1}^2 D = (-1)^{\ell m+t} Q_{\ell m+t}\, Q_0 = n.$$

Conversely, if Equation (1.8) is solvable for relatively prime natural numbers $x_0, y_0 \in \mathbb{N}$, then by [3, Theorem 6.2.7, p. 302], there exists a unique integer $P_0$ with $-|Q_0|/2 < P_0 \le |Q_0|/2$ and $P_0^2 \equiv D \pmod{|Q_0|}$ such that $Q_t = 1$ in the simple continued fraction expansion of $(P_0 + \sqrt{D})/Q_0$. By [3, Theorem 6.5.4, pp. 338-339], there exists a nonnegative integer $m$ such that $\ell m + t - 1$ is odd where $\ell = \ell((P_0 + \sqrt{D})/Q_0)$ with $x_0 = G_{\ell m+t-1}$ and $y_0 = B_{\ell m+t-1}$.

**Remark 3.2.** The solutions of Equation (3.13) can be broken down into equivalence classes as follows. First we use the symbols $(x_0, y_0)$ and $x_0 + y_0\sqrt{D}$ interchangeably to mean a solution of Equation (3.13). Two solutions $x_0 + y_0\sqrt{D}$ and $x_1 + y_1\sqrt{D}$ of Equation (3.13) are said to be in the same class provided that there exists a solution $u + v\sqrt{D}$ to the Pell equation:

$$x^2 - Dy^2 = 1, \tag{3.14}$$

such that

$$(x_0 + \sqrt{D}y_0)(u + v\sqrt{D}) = x_1 + y_1\sqrt{D}.$$

It can be shown (see [3, Proposition 6.2.1, p. 299]) that $x_0 + y_0\sqrt{D}$ and $x_1 + y_1\sqrt{D}$ are in the same class if and only if

$$\frac{x_0x_1 - Dy_0y_1}{n} \in \mathbb{Z} \quad \text{and} \quad \frac{x_0y_1 - y_0x_1}{n} \in \mathbb{Z}.$$

A solution $x_0 + y_0\sqrt{D}$ of (3.13) with $\gcd(x_0, y_0) = 1$ is called a *primitive solution*. Each primitive solution, as told to us by Theorem 3.3(a), determines a unique $P_0 \in \mathbb{Z}$ such that $-P_0y_0 \equiv x_0 \pmod{|n|}$ where $P_0^2 \equiv D \pmod{|n|}$ and $-|n|/2 < P_0 \le |n|/2$. The solution $x_0 + y_0\sqrt{D}$ is said to be belong to $P_0$ (see [3, Definition 6.2.4, p. 304]). There is a special type of solution class where both $x_0 + y_0\sqrt{D}$ and $-x_0 + y_0\sqrt{D}$ belong, in which case it is called *ambiguous*. For fundamental solutions of ambiguous classes, we assume that $x_0 \ge 0$.

A solution $x_0 + y_0\sqrt{D}$ in which $y_0$ is the least positive value for the class is called a *fundamental solution*. All solutions in a class with fundamental solution $x_0 + y_0\sqrt{D}$ are given by

$$\pm(u_0 + v_0\sqrt{D})^j(x_0 + y_0\sqrt{D}),$$

for some $j \in \mathbb{Z}$ where $u_0 + v_0\sqrt{D}$ is the fundamental solution of Pell's

Equation (3.14). It can be shown (see [3, pp. 299-301]) that there are only finitely many classes, so if $x_0 + y_0\sqrt{D}$ runs over all fundamental solutions of Equation (3.13), then

$$\pm\,(u_0 + v_0\sqrt{D})^j(x_0 + y_0\sqrt{D}), \text{ for all } j \in \mathbb{Z}$$

provide all solutions of Equation (3.13).

**Example 3.2.** Suppose that we want to determine solutions (if they exist) for $x^2 - 8y^2 = 17$, where $D = 8$ and $n = 17$. We first look at the solutions of the congruence $P_0^2 \equiv D \pmod{17}$ for $-17/2 = -n/2 < P_0 \le n/2 = 17/2$. These are $P_0 = \pm 5$. Forming $\alpha_0 = (P_0 + \sqrt{D})/Q_0 = (5 + \sqrt{8})/17$, we find that $Q_2 = 1 = Q_t$ in the simple continued fraction expansion of $\alpha_0$. Also, $\ell(\alpha_0) = \ell = 2$ and for $m = 0$, we get

$$x_0 = G_{\ell m+t-1} = G_1 = 7 \text{ and } y_0 = B_{\ell m+t-1} = B_1 = 2,$$

whence

$$x_0^2 - Dy_0^2 = 17 = n = Q_0 = 7^2 - 8\cdot 2^2.$$

Moreover, $7 + 2\sqrt{8}$ is a fundamental solution in its class. Since $3 + \sqrt{8}$ is a fundamental solution of Pell's Equation (3.14), then all solutions in its class are given by

$$\pm\,(3 + \sqrt{8})^j(7 + 2\sqrt{8}) \text{ for } j \in \mathbb{Z}. \tag{3.15}$$

If we look at the other possible value of $P_0$, namely $P_0 = -5$, then in the simple continued fraction expansion of $-\alpha' = (-5 + \sqrt{8})/17$, $t = 2 = \ell$ and for $m = 0$,

$$G_{\ell m+t-1} = G_1 = 5 \text{ and } B_{\ell m+t-1} = B_1 = 1,$$

so $5 + \sqrt{8}$ is a fundamental solution in its class and all solutions in its class are given by

$$\pm\,(3 + \sqrt{8})^j(5 + \sqrt{8}) \text{ for } j \in \mathbb{Z}. \tag{3.16}$$

Since no other possibilities exist for $P_0$, then (3.15)-(3.16) represent all solutions for $x^2 - 8y^2 = 17$.

On the other hand, if we are seeking a solution to $x^2 - 8y^2 = -17$, then we look again at the above congruence and get $P_0 = \pm 5$. However, here $\alpha_0 = (5+\sqrt{8})/(-17)$ has $Q_t = Q_3 = 1$ and $\ell = 2$, so $\ell m + t - 1 = 2m + 2$ which is never odd. Similarly, for $\alpha_0 = (-5+\sqrt{8})/(-17)$, $t = 1$ and $\ell = 2$, so $\ell m + t - 1 = 2m$ is never odd. Hence, Theorem 3.3 tells us that the latter equation has no solutions.

**Remark 3.3.** In [1], Matthews provides a simplified version of an algorithm for solving Equation (3.13), using simple continued fractions, first given by Lagrange in a memoir of 1770 [8, Oeuvres II, pp. 655-726]. In [1], Matthews asserts: "Lagrange's algorithm has been rediscovered by R. Mollin ([3, pp. 333-340]). His treatment is more complicated than ours, as it uses the language of ideals and semi-simple continued fractions". However, as we have seen in Theorem 3.3 and its illustration in Example 3.2, the proof given in [3] does indeed boil down to solely the use of simple continued fractions. In fact, Theorem 3.3 is a simpler version of the result in [1], although both versions are valuable applications of continued fractions.

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# CONTRACTIONS OF HARMONIC UNIVALENT FUNCTIONS

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## Abstract

We consider complex-valued harmonic functions of the form $f = h + \overline{g}$, where $h$ and $g$ are analytic in the open unit disk. For subclasses of convex or starlike harmonic functions, we investigate properties of the contraction map $f(cz)/c$, $0 < c < 1$. For these classes, we determine extreme points, sharp coefficient and distortion bounds and various containment properties.

## 1. Introduction

It is well-known that a function $f$ which is harmonic in the open unit disk $\Delta = \{z : |z| < 1\}$ can be written as $f = h + \overline{g}$ where $h$ and $g$ are analytic in $\Delta$. It is also known (see [4]) that the necessary and sufficient condition for the harmonic function $f = h + \overline{g}$ to be sense preserving and locally univalent in $\Delta$ is that the Jacobian $J_f$ is positive in $\Delta$, that is, if

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$|g'(z)| < |h'(z)|$ for $z \in \Delta$.

Let $\mathcal{HS}$ denote the family of functions $f = h + \overline{g}$ that are harmonic, complex-valued, orientation preserving, and univalent in the unit disk $\Delta$, with the normalization

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1}$$

We call $h$ the *analytic* and $g$ the *co-analytic part* of $f$. Note that $\mathcal{HS}$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions whenever the co-analytic part of $f$ is zero.

We may sometimes restrict ourselves to the subclass $\mathcal{H}^0\mathcal{S}$ of functions $f$ in $\mathcal{HS}$ so that $b_1 = f_{\overline{z}}(0) \equiv 0$. It is shown in [4] that both $\mathcal{HS}$ and $\mathcal{H}^0\mathcal{S}$ are normal families while only $\mathcal{H}^0\mathcal{S}$ is compact.

For $0 \le \alpha < 1$, let $\mathcal{HS}^*(\alpha)$ and $\mathcal{HK}(\alpha)$ consist of functions in $\mathcal{HS}$ which, respectively, are starlike of order $\alpha$ and convex of order $\alpha$ in $\Delta$.

The following two theorems are due to the second author and we shall need them throughout this paper.

**Theorem A** [5]. *Let $f = h + \overline{g}$ be so that $h$ and $g$ are given by (1). Then $f$ is harmonic starlike of order $\alpha$, denoted by $\mathcal{HS}^*(\alpha)$, if*

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} |b_n| \le 1, \quad 0 \le \alpha < 1. \tag{2}$$

*The harmonic starlike functions*

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1-\alpha}{n-\alpha} x_n z^n + \sum_{n=1}^{\infty} \frac{1-\alpha}{n+\alpha} \overline{y}_n \overline{z}^n, \tag{3}$$

*where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$, show that the bounds given by (2) is sharp.*

**Theorem B** [6]. *Let $f = h + \overline{g}$ be so that $h$ and $g$ are given by (1).*

*Then f is harmonic convex of order* $\alpha$, *denoted by* $\mathcal{HK}(\alpha)$, *if*

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha} |b_n| \le 1, \quad 0 \le \alpha < 1. \tag{4}$$

*The harmonic convex functions*

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1-\alpha}{n(n-\alpha)} x_n z^n + \sum_{n=1}^{\infty} \frac{1-\alpha}{n(n+\alpha)} \overline{y}_n \overline{z}^n, \tag{5}$$

*where* $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$, *show that the bounds given by* (4) *is sharp.*

Silverman [7] obtained results analogous to Theorems A and B for the special case $b_1 = \alpha = 0$ (also see [1]) and Silverman and Silvia [9] improved the results of [1] and [7] to the case $b_1$ not necessarily zero. In this paper, we extend the above results to contractions of the mappings in $\mathcal{HS}^*(\alpha)$ and $\mathcal{HK}(\alpha)$.

A function $F$ is said to be in $\mathcal{H}_c\mathcal{S}^*(\alpha)$ for some $c$, $0 \le c < 1$, if $F$ can be expressed by

$$F(z) = \frac{f(cz)}{c} = \frac{h(cz)}{c} + \frac{\overline{g(cz)}}{c} \tag{6}$$

for some $f = h + \overline{g}$, where $h$ and $g$ are the functions of the form (1) and $f$ satisfies the condition (2). Analogous to $\mathcal{H}_c\mathcal{S}^*(\alpha)$ is the family $\mathcal{H}_c\mathcal{K}(\alpha)$ consisting of functions $F$ that can be expressed as (6), where $f$ satisfies the condition (4).

Also, let $\mathcal{H}_c^0\mathcal{S}^*(\alpha)$ and $\mathcal{H}_c^0\mathcal{K}(\alpha)$ be the corresponding classes where $b_1 = 0$.

## 2. Inclusion Properties

For $f = h + \overline{g}$ as in Theorem A, we observe that the function $F \in \mathcal{H}_c\mathcal{S}^*(\alpha)$ may be written as

$$F(z) = z + \sum_{n=2}^{\infty} a_n c^{n-1} z^n + \sum_{n=1}^{\infty} c^{n-1} \overline{b_n} \overline{z}^n. \tag{7}$$

Therefore, if $0 < c_1 \le c_2 \le 1$, then $\mathcal{H}_{c_1}\mathcal{S}^*(\alpha) \subset \mathcal{H}_{c_2}\mathcal{S}^*(\alpha) \subset \mathcal{H}_1\mathcal{S}^*(\alpha) \equiv \mathcal{HS}^*(\alpha) \subset \mathcal{HS}$.

We remark that the families $\mathcal{H}_c\mathcal{S}^*(\alpha)$ and $\mathcal{H}_c\mathcal{K}(\alpha)$ are convex for $0 \le \alpha < 1$ and $0 \le c < 1$. This is an immediate consequence of the convexity for $c = 1$, which follows from (2), and the linearity of the operator $F(z) = f(cz)/c$.

**Theorem 1.** *$\mathcal{H}_c^0\mathcal{S}^*(\alpha) \subset \mathcal{H}^0\mathcal{K}(\beta)$ for*

$$c \le c_0 = \begin{cases} \inf_{2 \le n < \infty}\left[\left(\dfrac{n-\alpha}{1-\alpha}\right)\left(\dfrac{1-\beta}{n(n-\beta)}\right)\right]^{\frac{1}{n-1}}, & \text{if } 0 \le \alpha \le \beta < 1; \\ \inf_{2 \le n < \infty}\left[\left(\dfrac{n+\alpha}{1-\alpha}\right)\left(\dfrac{1-\beta}{n(n+\beta)}\right)\right]^{\frac{1}{n-1}}, & \text{if } 0 \le \beta < \alpha < 1. \end{cases} \tag{8}$$

*Theorem 1 is sharp for functions*

$$F_n(z) = z + \frac{1-\alpha}{n-\alpha} c^{n-1} z^n + \frac{1-\alpha}{n+\alpha} c^{n-1} \overline{z}^n, \quad (n = n(\alpha, \beta)).$$

**Proof.** In view of (4) and (7), it suffices to show that

$$\sum_{n=2}^{\infty} \frac{n(n-\beta)}{1-\beta} c^{n-1} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+\beta)}{1-\beta} c^{n-1} |b_n|$$

$$\le \sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} |b_n| \le 1.$$

This holds if $\dfrac{n(n-\beta)}{1-\beta} c^{n-1} \le \dfrac{n-\alpha}{1-\alpha}$ and $\dfrac{n(n+\beta)}{1-\beta} c^{n-1} \le \dfrac{n+\alpha}{1-\alpha}$ for each $n$. Solving these inequalities for $c$, we obtain the above required condition (8).

As a consequence of Theorem 1, we obtain the following corollaries.

**Corollary 1.** $\mathcal{H}_c^0\mathcal{S}^*(\alpha) \subset \mathcal{H}^0\mathcal{K}(\alpha)$ *for* $0 \le c \le 1/2$.

**Corollary 2.** $\mathcal{H}_c^0\mathcal{S}^*(0) \subset \mathcal{H}^0\mathcal{K}(\beta)$ *for* $0 \le c \le (1-\beta)/(2-\beta)$.

Further, if $a_k = b_k = 0,\ 2 \le k \le N,$ then we also have

**Corollary 3.** $\mathcal{H}_c^0\mathcal{S}^*(\alpha) \subset \mathcal{H}^0\mathcal{K}(\alpha)$ *for* $0 \le c \le \left(\dfrac{1}{N+1}\right)^{\frac{1}{N}}$.

**Corollary 4.** $\mathcal{H}_c^0\mathcal{S}^*(0) \subset \mathcal{H}^0\mathcal{K}(\beta)$ *for* $0 \le c \le \left(\dfrac{1-\beta}{N+1-\beta}\right)^{\frac{1}{N}}$.

These corollaries are a consequence of the fact that $\left(\dfrac{1-\beta}{n+1-\beta}\right)^{\frac{1}{n}}$ is an increasing function of $n$ for all $\beta \in [0, 1)$.

## 3. Extreme Points and Distortion Bounds

The determination of the extreme points of a compact family of univalent functions enables us to solve many extremal problems for the family. The fundamental reason for considering extreme points of convex hulls for starlike and convex is to more easily categorize extremal properties under continuous linear functionals acting on these classes. See [2] and [3]. We now determine the extreme points of the compact family $\mathcal{H}_c^0\mathcal{S}^*(\alpha)$. However, we first obtain the extreme points of the closed convex hull of $\mathcal{H}_1\mathcal{S}^*(\alpha)$ denoted by $clco\,\mathcal{H}_1\mathcal{S}^*(\alpha)$.

**Lemma 1.** *The extreme points of* $clco\,\mathcal{H}_1\mathcal{S}^*(\alpha)$ *consist of the functions* $h_1(z) = z,$

$$h_n(z) = z + \frac{1-\alpha}{n-\alpha}e^{i\alpha_n}z^n, \quad n = 2, 3, \ldots$$

*and*

$$g_n(z) = z + \frac{1-\alpha}{n+\alpha}e^{i\beta_n}z^n, \quad n = 1, 2, 3, \ldots.$$

**Proof.** It is sufficient to prove that $f \in clco\,\mathcal{H}_1\mathcal{S}^*(\alpha)$ if and only if $f$ can be expressed as $f(z) = \sum_{n=1}^{\infty}(\lambda_n h_n(z) + \mu_n g_n(z))$, where $\lambda_n \geq 0$, $\mu_n \geq 0$, and $\sum_{n=1}^{\infty}(\lambda_n + \mu_n) = 1$. Write

$$f(z) = \sum_{n=1}^{\infty}(\lambda_n h_n(z) + \mu_n g_n(z))$$

$$= z + \sum_{n=2}^{\infty}\frac{1-\alpha}{n-\alpha}e^{i\alpha_n}\lambda_n z^n + \sum_{n=1}^{\infty}\frac{1-\alpha}{n+\alpha}e^{-i\beta_n}\mu_n \bar{z}^n.$$

Then

$$\sum_{n=2}^{\infty}\frac{n-\alpha}{1-\alpha}\left(\frac{1-\alpha}{n-\alpha}\right)\left|e^{i\alpha_n}\right|\lambda_n + \sum_{n=1}^{\infty}\frac{n+\alpha}{1-\alpha}\left(\frac{1-\alpha}{n+\alpha}\right)\left|e^{i\beta_n}\right|\mu_n$$

$$= \sum_{n=2}^{\infty}\lambda_n + \sum_{n=1}^{\infty}\mu_n = 1 - \lambda_1 \leq 1.$$

Thus $f \in clco\,\mathcal{H}_1\mathcal{S}^*(\alpha)$. Conversely, suppose that $f \in clco\,\mathcal{H}_1\mathcal{S}^*(\alpha)$. From (2), it follows that $|a_n| \leq (1-\alpha)/(n-\alpha)$ and $|b_n| \leq (1-\alpha)/(n+\alpha)$ for all $n$. Therefore, we may set

$$\lambda_n = \frac{n-\alpha}{1-\alpha}|a_n| \quad (n = 2, 3, \ldots),$$

$$\mu_n = \frac{n+\alpha}{1-\alpha}|b_n| \quad (n = 1, 2, 3, \ldots),$$

and $\lambda_1 = 1 - \sum_{n=2}^{\infty}\lambda_n - \sum_{n=1}^{\infty}\mu_n$. Then

$$f(z) = z + \sum_{n=2}^{\infty}a_n z^n + \sum_{n=1}^{\infty}\bar{b}_n \bar{z}^n$$

$$= z + \sum_{n=2}^{\infty}|a_n|e^{i\alpha_n}z^n + \sum_{n=1}^{\infty}|b_n|e^{-i\beta_n}\bar{z}^n$$

$$= z + \sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{n-\alpha} e^{i\alpha_n} z^n + \sum_{n=1}^{\infty} \mu_n \frac{1-\alpha}{n+\alpha} e^{-i\beta_n} \overline{z}^n$$

$$= z + \sum_{n=2}^{\infty} \lambda_n (h_n(z) - z) + \sum_{n=1}^{\infty} \mu_n (g_n(z) - z)$$

$$= \sum_{n=1}^{\infty} (\lambda_n h_n(z) + \mu_n g_n(z)).$$

**Remark.** For $\alpha_n = \beta_n = \pi$, Lemma 1 gives the corresponding results obtained in [5].

**Theorem 2.** *The extreme points of* $\mathcal{H}_c^0 \mathcal{S}^*(\alpha)$ *consist of the functions* $P_1(z) = z$,

$$P_n(z) = z + \frac{1-\alpha}{n-\alpha} x c^{n-1} z^n, \quad |x| = 1 \quad (n = 2, 3, \ldots),$$

*and*

$$Q_n(z) = z + \frac{1-\alpha}{n+\alpha} \overline{y} c^{n-1} \overline{z}^n, \quad |y| = 1 \quad (n = 1, 2, 3, \ldots).$$

**Proof.** Consider the operator $\Lambda : \mathcal{H}_1^0 \mathcal{S}^*(\alpha) \to \mathcal{H}_c^0 \mathcal{S}^*(\alpha)$ defined by

$$\Lambda(f(z)) = F(z) = \frac{f(cz)}{c},$$

where $f = h + \overline{g}$ is in $\mathcal{H}_1^0 \mathcal{S}^*(\alpha)$. Note that $clco\, \mathcal{H}_1^0 \mathcal{S}^*(\alpha) = \mathcal{H}_1^0 \mathcal{S}^*(\alpha)$ and $\Lambda$ is an isomorphism from $\mathcal{H}_1^0 \mathcal{S}^*(\alpha)$ to $\mathcal{H}_c^0 \mathcal{S}^*(\alpha)$. Hence $\Lambda$ preserves extreme points and the result follows from Lemma 1.

Since the family $\mathcal{H}_c^0 \mathcal{S}^*(\alpha)$ is compact and convex, the maximum or minimum value on $\mathcal{H}_c^0 \mathcal{S}^*(\alpha)$ of the real part of any continuous linear functional occurs at one of the extreme points of $\mathcal{H}_c^0 \mathcal{S}^*(\alpha)$. Thus Theorem 2 yields the following two corollaries.

**Corollary 5.** *If*

$$F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=2}^{\infty} \overline{B}_n \overline{z}^n \in \mathcal{H}_c^0 \mathcal{S}^*(\alpha),$$

*then* $|A_n| \leq \dfrac{1-\alpha}{n-\alpha} c^{n-1}$ *and* $|B_n| \leq \dfrac{1-\alpha}{n+\alpha} c^{n-1}$. *These bounds are sharp for*

$$F(z) = z + \frac{1-\alpha}{n-\alpha} c^{n-1} z^n + \frac{1-\alpha}{n+\alpha} c^{n-1} \overline{z}^n \in \mathcal{H}_c^0 \mathcal{S}^*(\alpha).$$

**Corollary 6.** *If* $F \in \mathcal{H}_c^0 \mathcal{S}^*(\alpha)$, *then*

$$r - \frac{1-\alpha}{2+\alpha} cr^2 \leq |F(z)| \leq r + \frac{1-\alpha}{2-\alpha} cr^2 \quad (|z| = r < 1).$$

*Equality holds for functions*

$$F(z) = z + \frac{1-\alpha}{2-\alpha} cz^2$$

*and*

$$F(z) = z + \frac{1-\alpha}{2+\alpha} c\overline{z}^2.$$

Our next theorem can be proved by using similar argument to that used to prove Theorem 2.

**Theorem 3.** *The extreme points of* $\mathcal{H}_c^0 \mathcal{K}(\alpha)$ *consists of the functions given by* $P_1(z) = z$,

$$P_n(z) = z + \frac{1-\alpha}{n(n-\alpha)} xc^{n-1} z^n, \quad |x| = 1 \quad (n = 2, 3, \ldots),$$

*and*

$$Q_n(z) = z + \frac{1-\alpha}{n(n+\alpha)} \overline{y} c^{n-1} \overline{z}^n, \quad |y| = 1 \quad (n = 1, 2, 3, \ldots).$$

Similarly, we obtain the following corollaries.

**Corollary 7.** *If*

$$F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=2}^{\infty} \overline{B}_n (\overline{z})^n \in \mathcal{H}_c^0\mathcal{K}(\alpha),$$

*then* $|A_n| \le \dfrac{1-\alpha}{n(n-\alpha)} c^{n-1}$ *and* $|B_n| \le \dfrac{1-\alpha}{n(n+\alpha)} c^{n-1}$. *These bounds are sharp for*

$$F(z) = z + \frac{1-\alpha}{n(n-\alpha)} c^{n-1} z^n + \frac{1-\alpha}{n(n+\alpha)} c^{n-1} \overline{z}^n.$$

**Corollary 8.** *If* $F \in \mathcal{H}_c^0\mathcal{K}(\alpha)$, *then*

$$r - \frac{1-\alpha}{2(2+\alpha)} cr^2 \le |F(z)| \le r + \frac{1-\alpha}{2(2-\alpha)} cr^2 \quad (|z| = r < 1).$$

*Equality holds for functions*

$$F(z) = z + \frac{1-\alpha}{2(2-\alpha)} cz^2$$

*and*

$$F(z) = z + \frac{1-\alpha}{2(2+\alpha)} c\overline{z}^2.$$

Next, we determine the distortion bounds for functions in $\mathcal{H}_c\mathcal{S}^*(\alpha)$.

**Theorem 4.** *If* $F \in \mathcal{H}_c\mathcal{S}^*(\alpha)$ *and* $(1-\alpha)/(1+\alpha) \le |b_1| < 1$, *then for some* $0 < c \le 1$ *and* $|z| = r < 1$,

$$(1 - |b_1|)r - \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right)cr^2$$

$$\le |F(z)| \le (1 + |b_1|)r + \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right)cr^2.$$

**Proof.** We only prove the right hand bound. The proof for the left hand bound is similar and we omit it. For $F$ of the form (7), we have

$$|F(z)| \le (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)c^{n-1}r^n$$

$$\le (1+|b_1|)r + \frac{1-\alpha}{2-\alpha}\sum_{n=2}^{\infty}\left(\frac{2-\alpha}{1-\alpha}|a_n| + \frac{2-\alpha}{1-\alpha}|b_n|\right)cr^2$$

$$\le (1+|b_1|)r + \frac{1-\alpha}{2-\alpha}\sum_{n=2}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right)cr^2$$

$$\le (1+|b_1|)r + \frac{1-\alpha}{2-\alpha}\left(1 - \frac{1+\alpha}{1-\alpha}|b_1|\right)cr^2$$

$$= (1+|b_1|)r + \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right)cr^2.$$

This bound is sharp for functions

$$F(z) = z + |b_1|\bar{z} + \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right)c\bar{z}^2.$$

A similar argument yields the distortion bounds for functions in $\mathcal{H}_c\mathcal{K}(\alpha)$.

**Theorem 5.** *If* $F \in \mathcal{H}_c\mathcal{K}(\alpha)$ *and* $(1-\alpha)/(1+\alpha) \le |b_1| < 1$, *then for some* $0 < c \le 1$ *and* $|z| = r < 1$,

$$(1-|b_1|)r - \frac{1}{2}\left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right)cr^2$$

$$\le |F(z)| \le (1+|b_1|)r + \frac{1}{2}\left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right)cr^2.$$

## 4. Starlikeness and Convexity Properties

**Theorem 6.** *For* $c$, $\alpha$, $\beta$ *in the interval* $[0, 1)$, *there exists a function* $F \in \mathcal{HS}^*(\beta) - \mathcal{H}_c\mathcal{S}^*(\alpha)$.

**Proof.** The function $f_n(z) = z + \frac{1-\beta}{n-\beta}z^n$ is in $\mathcal{HS}^*(\beta)$ for every $n$. But $f_n(z)$ is in $\mathcal{H}_c\mathcal{S}^*(\alpha)$ only when $\frac{1-\beta}{n-\beta} \le \frac{1-\alpha}{n-\alpha}c^{n-1}$ or equivalently,

$$\frac{(1-\alpha)(n-\beta)}{(1-\beta)(n-\alpha)}c^{n-1} \geq 1. \tag{9}$$

Since the left hand side of (9) approaches 0 as $n \to \infty$, $f_n \in \mathcal{HS}^*(\beta) - \mathcal{H}_c\mathcal{S}^*(\alpha)$ for $n = n(c, \alpha, \beta)$ sufficiently large.

**Remark.** The above theorem shows that $\mathcal{H}^0\mathcal{S}^*(\beta) \not\subset \mathcal{H}_c^0\mathcal{S}^*(\alpha)$.

In the next theorem we determine $\alpha$ and $\beta$ so that the reverse inclusion holds.

**Theorem 7.** *Let* $0 < c \leq 1$, $0 \leq \alpha < 1$. *Then* $\mathcal{H}_c^0\mathcal{S}^*(\alpha) \subset \mathcal{H}^0\mathcal{S}^*(\beta)$ *if* $\beta = \dfrac{2-\alpha-2(1-\alpha)c}{2-\alpha-(1-\alpha)c}$. *The result is sharp with the extreme functions*

$$F(z) = z + \frac{1-\alpha}{2-\alpha}e^{i\theta}cz^2 + \frac{1-\alpha}{2+\alpha}e^{i\phi}c\bar{z}^2$$

*and*

$$F(z) = z + \frac{1-\beta}{2-\beta}e^{i\theta}cz^2 + \frac{1-\beta}{2+\beta}e^{i\phi}c\bar{z}^2.$$

**Proof.** We need to show that if

$$\sum_{n=2}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|\,a_n\,| + \frac{n+\alpha}{1-\alpha}|\,b_n\,|\right) \leq 1,$$

then

$$\sum_{n=2}^{\infty}\left(\frac{n-\beta}{1-\beta}c^{n-1}|\,a_n\,| + \frac{n+\beta}{1-\beta}c^{n-1}|\,b_n\,|\right) \leq 1.$$

It suffices to prove that both inequalities $\dfrac{1-\alpha}{n-\alpha}c^{n-1} \leq \dfrac{1-\beta}{n-\beta}$, $\dfrac{1-\alpha}{n+\alpha}c^{n-1} \leq \dfrac{1-\beta}{n+\beta}$ hold for all integers $n \geq 2$. Equivalently, we must show, for every $n$, that

$$\zeta_1(n) = \left(\frac{n-\beta}{1-\beta}\right)\left(\frac{1-\alpha}{n-\alpha}\right)c^{n-1} \leq 1 \tag{10}$$

and

$$\zeta_2(n) = \left(\frac{n+\beta}{1-\beta}\right)\left(\frac{1-\alpha}{n+\alpha}\right)c^{n-1} \le 1. \tag{11}$$

For (10), since

$$\zeta_1(2) = \left(\frac{2-\alpha}{c(1-\alpha)}\right)\left(\frac{1-\alpha}{2-\alpha}\right)c = 1,$$

we need to show that if $\zeta_1(n) \le 1$, then $\zeta_1(n+1) \le 1$, that is, if $(n-\beta)(1-\alpha)c^{n-1} \le (n-\alpha)(1-\beta)$, then $(n+1-\beta)(1-\alpha)c^n \le (n+1-\alpha)(1-\beta)$. By induction

$$\begin{aligned}(n+1-\beta)(1-\alpha)c^n &= ((n-\beta)(1-\alpha)c^{n-1})c + (1-\alpha)c^n \\ &\le (n-\alpha)(1-\beta)c + (1-\alpha)c^n \\ &\le (n-\alpha)(1-\beta) + (1-\alpha)c^2 \\ &\le (n+1-\alpha)(1-\beta)\end{aligned}$$

provided that $(1-\alpha)c^2 \le 1-\beta = (1-\alpha)c/((2-\alpha)-(1-\alpha)c)$ or provided that $(1-\alpha)c^2 - (2-\alpha)c + 1 \ge 0$, which holds if $c \le 1$. This proves (10).

Next we show that (11) holds for all $n$ and

$$\beta^* = [2+\alpha-2(1-\alpha)c]/[(1-\alpha)c+2+\alpha].$$

First note that $\beta^* \le \beta$ because

$$\beta - \beta^* = \frac{4(1-\alpha)^2 c(1-c)}{(2-\alpha-(1-\alpha)c)((1-\alpha)c+2+\alpha)} \ge 0$$

for some $c(0 < c \le 1)$. Since $\zeta_2(2) = 1$ for $\beta = \beta^*$, it is only left to show that if $\zeta_2(n) \le 1$, then $\zeta_2(n+1) \le 1$ for $\beta = \beta^*$. Or equivalently, if $(n+\beta^*)(1-\alpha)c^{n-1} \le (1-\beta^*)(n+\alpha)$, then $(n+1+\beta^*)(1-\alpha)c^n \le (1-\beta^*)(n+1+\alpha)$. By induction

$$
\begin{aligned}
(n+1+\beta^*)(1-\alpha)c^n &= ((n+\beta^*)(1-\alpha)c^{n-1})c+(1-\alpha)c^n \\
&\le (n+\alpha)(1-\beta^*)c+(1-\alpha)c^n \\
&\le (n+\alpha)(1-\beta^*)+(1-\alpha)c^2 \\
&\le (n+1+\alpha)(1-\beta^*)
\end{aligned}
$$

provided that

$$(1-\alpha)c^2 \le 1-\beta^* = \frac{3(1-\alpha)c}{2+\alpha+(1-\alpha)c},$$

which holds when $(1-\alpha)c^2+(2+\alpha)c-3 \le 0$ or if $(c-1)(3+c(1-\alpha)) \le 0$. But this last inequality holds for $c \le 1$. This completes the proof.

The proof for the following theorem is similar to that given above and will be omitted.

**Theorem 8.** *With the above notations, we have* $\mathcal{H}_c^0\mathcal{K}(\alpha) \subset \mathcal{H}^0\mathcal{K}(\beta)$ *with the extremal functions*

$$F(z) = z + \frac{1-\alpha}{2(2-\alpha)}e^{i\theta}cz^2 + \frac{1-\alpha}{2(2+\alpha)}e^{i\phi}c\bar{z}^2$$

*and*

$$F(z) = z + \frac{1-\beta}{2(2-\beta)}e^{i\theta}cz^2 + \frac{1-\beta}{2(2+\beta)}e^{i\phi}c\bar{z}^2.$$

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# GLOBALLY ASYMPTOTIC BEHAVIORS OF SOLUTIONS FOR AN INTEGRO-DIFFERENTIAL EQUATION WITH INFINITE DELAY

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## Abstract

By using the stability criteria in a Banach space $BC$ rather than in an admissible space, the globally asymptotic behaviors for two models of population dynamics in hematology are discussed, and the sufficient conditions for global stability of equilibria of the models are obtained.

## 1. Introduction

Consider the following equations

$$\frac{dN(t)}{dt} = -\gamma N(t) + \alpha \int_0^\infty K(s)\, F(N(t-s))\, ds, \quad t \geq 0, \tag{1.1}$$

and

$$\frac{dN(t)}{dt} = -\gamma N(t) - \beta F(N(t)) + \alpha \int_0^\infty K(s)\, F(N(t-s))\, ds, \quad t \geq 0, \tag{1.2}$$

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where

$$F(u) = \frac{u}{1 + u^n}, \qquad n \in (0, 1], \tag{1.3}$$

and the parameters $\gamma > 0$, $\alpha > 0$, $\beta > 0$, $n \in (0, 1]$; $K : [0, \infty) \to [0, \infty)$ is integrable and

$$\int_0^\infty K(s)\,ds = 1. \tag{1.4}$$

If $K(s) = \delta(s - \tau)$, $\delta$ is the Dirac Delta function, then (1.1) and (1.2) become the situation of finite delay which are the models of population dynamics in hematology posed by Mackey et al. [6-10], where $N(t)$ denotes the density of mature stem cells in the blood circulatory procedure, and $\tau$ is the cell circle time in the proliferating phase. Thus Eqs. (1.1) and (1.2) are the generalizations of the above models while the delay is continuously distributed on $[0, \infty)$. They are functional differential equations (resp. FDE) with infinite delays.

For convenience, we introduce the following notations:

$C = C[(-\infty, 0], R]$, $BC = (C, \|\cdot\|) = \{\phi \in C : \|\phi\| = \sup_{-\infty < s \le 0} |\phi(s)| < \infty\}$,

$G = \{g \in C[(-\infty, 0], [1, \infty)] : g(0) = 1, \lim_{s \to -\infty} g(s) = +\infty,$ $g(s)$ is nonincreasing$\}$

$$C_g = (C_g, \|\cdot\|_g) = \left\{\phi \in C : \|\phi\|_g = \sup_{-\infty < s \le 0}\left|\frac{\phi(s)}{g(s)}\right| < \infty\right\}, \text{ where } g \in G,$$

$$B(r) = \{\phi \in BC : \|\phi\| < r\}, \qquad B_g(r) = \{\phi \in C_g : \|\phi\|_g < r\},$$

$$UC_g = \{\phi \in C_g : \frac{\phi}{g} \text{ is uniformly continuous on } (-\infty, 0]\}, \text{ where } g \in G$$

and $\lim_{u \to 0-} \dfrac{g(s+u)}{g(s)} = 1$ holds uniformly on $(-\infty, 0]$.

$BC$ and $C_g$ are both Banach spaces [2]. As we know, the basic theory (including existence, uniqueness, continuous dependence, continuation) of FDE with infinite delays are established in an admissible space [3, 5].

Generally speaking, there are many choices of admissible spaces, and $UC_g$ is one of them [1, 2]. Note that $BC$ is not an admissible space, and the positive orbit of bounded solution for the equations is not always precompact in $BC$. In [4], Gopalsamy and Weng discussed the global attractivity and oscillations of solutions for the Eq. (1.1) in $BC$ with

$$F(u) = \frac{1}{1 + u^n}, \qquad n \geq 1.$$

But they did not involve the basic theory of Eq. (1.1). Recently, Feng [3] used the concepts of fading memory and integral fading memory to discuss the autonomous FDE

$$\dot{x}(t) = f(x_t) \tag{1.5}$$

with infinite delays so that the basic properties of which can locally be extended from $BC$ to $C_g$, and then he gave a criterion for globally asymptotic behaviors of (1.5) which can be examined in $BC$. Note that $f(0) = 0$ and this implies that (1.5) has the trivial solution. In this article, we shall show that (1.1) and (1.2) satisfy the conditions in this criterion under some choice of the parameters and the integral kernel $K$. Thus the equilibria of (1.1) and (1.2) are globally asymptotically stable.

In view of the actual background of Eqs. (1.1) and (1.2), it can be understood that every solution with positive initial condition keeps positive on its existing interval. Thus we only consider $F(u)$ for $u \geq 0$.

## 2. Functional with Fading Memory

Firstly, we introduce the following definitions and lemmas:

**Definition 2.1** [3]. A functional $f : [0, \infty) \times BC \to R$ is said to be *fading memory locally relative to* $g$ (resp. FMLR-$g$), if for every $r > 0$, there is a $g \in G$ such that $f$ is defined in $[0, \infty) \times B_g(r)$, and for $\varepsilon > 0$, $T > 0$, there exists $k > 0$ such that $\{\phi, \varphi \in B_g(r), \phi(s) = \varphi(s), s \in [-k, 0]\}$ implies that

$$|f(t, \phi) - f(t, \varphi)| < \varepsilon \quad \text{for } t \in [0, T].$$

**Definition 2.2** [1, 3]. Functional $Q : [0, \infty) \times R \times R \to R$ is said to be *integral fading memory* if for any $\varepsilon > 0$ and $r > 0$, there exists $k > 0$ such that

$$\int_{-\infty}^{-k} Q(t, t+s, \phi(s))\, ds < \varepsilon$$

for all $t \geq 0$ and all $\phi \in BC$ with $\| \phi \| \leq r$.

**Lemma 2.1** [1, 3]. *If $Q$ is continuous and has integral fading memory, then for every $r > 0$, there is a $g \in G$ such that*

$$\int_{-\infty}^{-k} | Q(t, t+s, \phi(s)) |\, ds \to 0, \quad as \quad k \to +\infty$$

*uniformly for $t \geq 0$ and $\phi \in C_g$ with $\| \phi \|_g \leq r$.*

**Remark 1.1.** We can see from Definitions 2.1-2.2 and Lemma 2.1 that if $Q$ is continuous and has integral fading memory, then

$$F(t, \phi) = \int_{-\infty}^{0} Q(t, t+s, \phi(s))\, ds$$

has FMLR-$g$.

**Remark 1.2.** If $g_1 \in G$ satisfies the conditions in Definition 2.1, then every $g \in G$ satisfying $g(t) \leq g_1(t)$ for all $t \in (-\infty, 0]$ can play the same role as $g_1(t)$.

**Lemma 2.2** [3]. *Assume that there exists a functional $V : BC \to R$ satisfying*

1, *$V$ is continuous in $BC$, and has FMLR-$g$;*

2. *there is a nondecreasing and nonnegative continuous function $u(r)$ such that $u(r) \to \infty$ as $r \to \infty$, and $u(| \phi(0) |) \leq V(\phi)$ for every $\phi \in BC$;*

3. *there is a nonnegative continuous function $W$ such that $W(r) > 0$, $r \neq 0$, and $\dot{V}_{(1.5)}(\phi) \leq -W(| \phi(0) |)$ for all $\phi \in BC$.*

*Then the zero solution of (1.5) is globally asymptotically stable in $BC$.*

Define $f : BC \to R$ as follows:

$$f(\phi) = -\gamma\phi(0) + \alpha \int_{-\infty}^{0} K(-s)\, F(\phi(s))\, ds,$$

or

$$f(\phi) = -\gamma\phi(0) - \beta F(\phi(0)) + \alpha \int_{-\infty}^{0} K(-s)\, F(\phi(s))\, ds.$$

Then (1.1) and (1.2) can be written as

$$\frac{dN(t)}{dt} = f(N_t), \quad t \geq 0, \tag{2.1}$$

where $N_t(\theta) = N(t + \theta), -\infty < \theta \leq 0$. (2.1) is an autonomous FDE. The positive equilibrium $N = N^*$ of it is the positive solution of the Eq.

$$\gamma u = \alpha F(u) \quad \text{or} \quad \gamma u = (\alpha - \beta)\, F(u). \tag{2.2}$$

The following are obtained by analysing the properties of $F(u)$:

1. (2.1) has equilibria $N = 0$ and $N = N^* > 0$ if $\gamma < \alpha$ (corresponding to (1.1)) or $\gamma < \alpha - \beta$ (corresponding to (1.2));

2. (2.1) has only zero equilibrium $N = 0$ if $\gamma \geq \alpha$ (corresponding to (1.1)) or $\gamma \geq \alpha - \beta > 0$ (corresponding to (1.2)).

Suppose $N^*$ is the positive equilibrium of (2.1). Let $y = N - N^*$. Define a functional $H : BC \to R$ as follows:

$$H(\phi) = -\gamma\phi(0) + \alpha \int_{-\infty}^{0} K(-s)[F(N^* + \phi(s)) - F(N^*)]\, ds,$$

or

$$H(\phi) = -\gamma\phi(0) - \beta[F(N^* + \phi(0)) - F(N^*)]$$
$$+ \alpha \int_{-\infty}^{0} K(-s)[F(N^* + \phi(s)) - F(N^*)]\, ds.$$

Then (2.1) is transformed into

$$\frac{dy(t)}{dt} = H(y_t), \quad t \geq 0. \tag{2.3}$$

We assert that $H(\phi)$ satisfies the following conditions (but we omit the details of verification):

$(\mathrm{A}_1)$ $H(0) = 0$;

$(\mathrm{A}_2)$ $H$ has FMLR-$g$;

$(\mathrm{A}_3)$ $H(\phi)$ satisfies the local Lipschitz condition in $BC$.

Thus we obtain from $(\mathrm{A}_1)$-$(\mathrm{A}_3)$ that for every $r > 0$, there exists a $g \in G$ such that the existence-uniqueness-continuous dependence-continuation of solutions for Eq. (2.3) hold in $B_g(r)$ [3]. The same conclusion holds for Eq. (2.1).

Assume that $N(t, \phi)$ is the solution of (2.1) through $(0, \phi)$ defined on $[0, \infty)$. Since (2.1) is an autonomous system, $N_t$ possesses the semi-group property, that is $N_t(N_s(\phi)) = N_{t+s}(\phi)$ for any $t \geq 0$, $s \geq 0$. We shall use this conclusion in Section 4.

## 3. Estimate of Solutions

For the convenience of use in Section 4, we give a lower bound estimate for solutions of (2.1). Since procedure for proof is similar, we only give the outlines for (1.2). Firstly, we state the properties of function $F(u) = \dfrac{u}{1+u^n}$ for $n \in (0, 1]$:

(P) $F(u)$ is increasing for $u > 0$, $\lim\limits_{u\to\infty} F(u) = 1$ as $n = 1$, and $\lim\limits_{u\to\infty} F(u) = +\infty$ as $n \in (0, 1)$.

Consider another function

$$g(u) = -(\gamma + \beta) + \frac{\alpha}{1 + u^n}, \qquad u \geq 0. \tag{3.1}$$

Suppose that $\alpha > \gamma + \beta$, then we have

$$g(A) = 0, \;\; A = \left(\frac{\alpha}{\gamma + \beta} - 1\right)^{\frac{1}{n}}; \;\; g(u) > 0, \;\; u \in (0, A). \tag{3.2}$$

**Lemma 3.1.** *Assume that* $N(t)$ *is any positive solution of* (1.2) *and* $\alpha - \beta > \gamma$. *For any* $b \in (0, 1)$, *there exists* $T_1 > 0$ *such that*

$$N(t) \geq N_0 \overset{\Delta}{=} bA, \quad t \geq T_1. \tag{3.3}$$

**Proof.** Let $N(t)$ be any positive solution of (1.2). If (3.3) is not true, then there are two possibilities:

(I) there is $T_2 > 0$, such that $N(t) < N_0$ for $t \geq T_2$;

(II) $N(t)$ is oscillatory about $N_0$.

Noting that $N_0 \in (0, A)$, from (3.2), we have

$$g(N_0) = -(\gamma + \beta) + \frac{\alpha}{1 + N_0^n} > 0. \tag{3.4}$$

Suppose that (I) is true. By using (1.4) and (3.4), one can choose a $\sigma > 0$ such that

$$0 < \eta \overset{\Delta}{=} \int_0^\sigma K(s)\, ds \leq 1, \qquad -(\gamma + \beta) + \frac{\alpha\eta}{1 + N_0^n} > 0. \tag{3.5}$$

We derive from (1.2) the positivity of $N(t)$ that

$$\frac{dN(t)}{dt} \geq -(\gamma + \beta)\, N(t) + \alpha \int_0^\sigma K(s)\, F(N(t - s))\, ds. \tag{3.6}$$

Under (I), there are three sub-cases:

($1^o$) $N(t)$ is decreasing for $t \geq T_3 \geq T_2$;

($2^o$) there is a sequence $\{t_k\}$ such that $t_k > 0$, $\lim_{k\to\infty} t_k = +\infty$, and $N(t_k)$ $(k = 1, 2, \ldots)$ are local minima of $N(t)$;

($3^o$) $N(t)$ is increasing for $t \geq T_4 \geq T_2$.

We first say that ($1^o$) is impossible. Otherwise, we obtain from (3.5) and (3.6) that

$$\frac{dN(t)}{dt} \geq N(t)\left[-(\gamma+\beta)+\frac{\alpha\eta}{1+N_0^n}\right] > 0, \quad \text{for } t \geq T_3+\sigma,$$

which is a contradiction to $(1^o)$.

Secondly, we investigate $(2^o)$. Let

$$B = \liminf_{t\to\infty} N(t) \geq 0.$$

Then $B > 0$. Otherwise $B = 0$. In view of the positivity of $N(t)$ and the definition of lower limit, one can choose a subsequence $\{t_{k_q}\} \subset \{t_k\}$ such that

$$N(t_{k_q}) = \min\{N(t) \mid 0 \leq t \leq t_{k_q}\}, \quad \lim_{q\to\infty} N(t_{k_q}) = 0.$$

Choose some $t_{k_m} \in \{t_{k_q}\}$ such that $t_{k_m} \geq T_2 + \sigma$,

$$\begin{aligned}
0 = \frac{dN(t)}{dt}\Big|_{t=t_{k_m}} &\geq -(\gamma+\beta)N(t_{k_m}) + \alpha\int_0^\sigma K(s)\,F(N(t_{k_m}-s))\,ds \\
&\geq -(\gamma+\beta)N(t_{k_m}) + \alpha\int_0^\sigma K(s)\frac{N(t_{k_m}-s)}{1+N_0^n}\,ds \\
&\geq N(t_{k_m})\left[-(\gamma+\beta)+\frac{\alpha\eta}{1+N_0^n}\right] > 0. \qquad (3.7)
\end{aligned}$$

The contradiction in (3.7) says that $B > 0$. Thus we have from (I) that

$$0 < B = \liminf_{t\to\infty} N(t) = \liminf_{k\to\infty} N(t_k) \leq N_0. \tag{3.8}$$

Noting that

$$-(\gamma+\beta)B + \alpha\eta F(B) \geq B\left[-(\gamma+\beta)+\frac{\alpha\eta}{1+N_0^n}\right] > 0 \tag{3.9}$$

and the continuity of $F(u)$, we could choose $\varepsilon_0 > 0$ small enough such that

$$-(\gamma+\beta)(B+\varepsilon_0) + \alpha\eta F(B-\varepsilon_0) > 0. \tag{3.10}$$

For such $\varepsilon_0$, by using the definition of lower limit and (3.8), we know that

there exist $T_5 > T_2$ and subsequence $\{t_{k_j}\} \subset \{t_k\}$ such that

$$N(t) > B - \varepsilon_0 \quad \text{for} \quad t \geq T_5,\ N(t_{k_j}) < B + \varepsilon_0, \quad j = 1, 2, \ldots. \tag{3.11}$$

Note that

$$B - \varepsilon_0 < N(t) \leq N_0 \quad \text{for } t \geq T_5.$$

Since $F(u)$ is increasing for $u$, in view of (3.6) and (3.10)-(3.11), we obtain

$$0 = \frac{dN(t)}{dt}\Big|_{t=t_{k_j}} \geq -(\gamma + \beta)\, N(t_{k_j}) + \alpha \int_0^\sigma K(s)\, F(N(t_{k_j} - s))\, ds$$

$$\geq -(\gamma + \beta)(B + \varepsilon_0) + \alpha\eta F(B - \varepsilon_0) > 0 \tag{3.12}$$

as $j$ is large enough. This is a contradiction, thus $(2^o)$ is impossible.

If $(3^o)$ holds, i.e., $N(t)$ is increasing eventually, then $\lim\limits_{t\to\infty} N(t)$ exists and

$$0 < C = \lim_{t\to\infty} N(t) \leq N_0. \tag{3.13}$$

By a similar deriving procedure as in (3.9)-(3.10), one could select a small $\varepsilon_1 > 0$ such that $C - \varepsilon_1 > 0$ and

$$-(\gamma + \beta)C + \frac{\alpha\eta(C - \varepsilon_1)}{1 + N_0^n} > 0. \tag{3.14}$$

Due to the monotonicity of $N(t)$, there is a $T_6 \geq T_2$ such that

$$C - \varepsilon_1 < N(t) \leq C \leq N_0 \quad \text{for } t \geq T_6. \tag{3.15}$$

We derive from (3.6) and (3.14) that

$$\frac{dN(t)}{dt} \geq -(\gamma + \beta)\, N(t) + \alpha \int_0^\sigma K(s) \frac{N(t - s)}{1 + N_0^n}\, ds$$

$$\geq -(\gamma + \beta)C + \frac{\alpha\eta(C - \varepsilon_1)}{1 + N_0^n} > 0$$

for $t \geq T_6 + \sigma$, which leads to $\lim\limits_{t\to\infty} N(t) = +\infty$. This is impossible. Thus $(3^o)$ is false. In summary, (I) is not true.

Now consider the case (II). Suppose that there exists a sequence $\{t_k\}$ such that

$$\lim_{k\to\infty} t_k = +\infty, \quad N(t_k) \le N_0 \ (k = 1, 2, \ldots), \tag{3.16}$$

and $N(t_k)$ $(k = 1, 2, \ldots)$ are local minima of $N(t)$. Furthermore, we have

$$E = \liminf_{t\to\infty} N(t) = \liminf_{k\to\infty} N(t_k) \le N_0.$$

One can show that $E > 0$. If $E = 0$, then by a similar procedure as in case (I), one can find some $t_{k_m} \in \{t_k\}$ such that

$$t_{k_m} - \sigma \ge 0, \quad N(t_{k_m}) = \min\{N(t) \mid 0 \le t \le t_{k_m}\} \le N_0. \tag{3.17}$$

For any $s \in [0, \sigma]$, if $N(t_{k_m} - s) \ge N_0$, noting the monotonicity of $F(u)$, we have

$$F(N(t_{k_m} - s)) \ge F(N_0) \ge \frac{N(t_{k_m})}{1 + N_0^n};$$

if $N(t_{k_m} - s) < N_0$. Then we have

$$F(N(t_{k_m} - s)) \ge \frac{N(t_{k_m} - s)}{1 + N_0^n} \ge \frac{N(t_{k_m})}{1 + N_0^n}.$$

However, we always have

$$F(N(t_{k_m} - s)) \ge \frac{N(t_{k_m})}{1 + N_0^n} \quad \text{for} \quad s \in [0, \sigma]. \tag{3.18}$$

Use (3.18) to get (3.7). Thus $E > 0$.

Finally, by replacing $B$, $T_5$ in (3.8)-(3.12) with $E$, $T_7$ ($T_7$ is large enough), we also obtain a conclusion that (II) is impossible. Thus we complete the proof.

The following is the similar conclusion about Eq. (1.1).

**Lemma 3.2.** *Assume that $N(t)$ is any positive solution of (1.1) and $\alpha > \gamma$. Then for any $b \in (0, 1)$, there is a $T_1 > 0$ such that*

$$N(t) \ge N_0 \triangleq bA, \quad t \ge T_1.$$

## 4. Globally Asymptotic Stability

In this section, we discuss Eq. (2.1) in the region

$$BC_+ = \{\phi \in BC : \phi(s) \geq 0,\ s \in (-\infty, 0]\}$$

or

$$BC_* = \{\phi \in BC_+ : \phi(0) > 0\}.$$

We shall only show the conclusions for Eq. (1.2) in all situations. First, we give the following lemma without proof.

**Lemma 4.1.** *The solution* $N(t)$ *of the initial problem*

$$\frac{dN(t)}{dt} = f(N_t), \quad t \geq 0, \quad N_t|_{t=0} = \phi \in BC_* \tag{4.1}$$

*satisfies* $N_t \in BC_*$ *for all* $t > 0$.

The following theorem is about the stability of the zero solution of Eq. (2.1).

**Theorem 4.1.** *Assume* $\gamma > \alpha$ *(for Eq.* (1.1)) *or* $\gamma > \alpha - \beta$ *(for Eq.* (1.2)), *and*

$$\int_0^\infty sK(s)\,ds < \infty. \tag{4.2}$$

*Then the zero solution of* (2.1) *is globally asymptotically stable in* $BC_+$. *The attractive region is* $BC_+$.

**Proof.** Let a functional $V(\phi)$ be defined by

$$V(\phi) = \phi(0) + \alpha \int_{-\infty}^0 \left( \int_{-s}^\infty K(u)\,du \right) F(\phi(s))\,ds, \quad \text{for } \phi \in BC_+.$$

For $\varepsilon > 0$, $r > 0$, since $F(u)$ is increasing for $u > 0$, if $\|\phi\| \leq r$, then we have $|F(\phi(s))| \leq F(\|\phi\|) \leq F(r)$, and thus in view of (4.2), there is a $k > 0$ such that

$$\left| \int_{-\infty}^{-k} \left( \int_{-s}^\infty K(u)\,du \right) F(\phi(s))\,ds \right| \leq |F(r)| \int_k^\infty sK(s)\,ds < \varepsilon,$$

which implies that $V(\phi)$ has FMLR-$g$.

Note that when $\phi(s) = N_t(s)$, we have

$$V(N_t) = N(t) + \alpha \int_0^{\infty} K(s) \left( \int_{t-s}^{t} F(N(u))\, du \right) ds.$$

Now, we calculate the rate of change of $V$ along the solution $N(t)$ of (1.2), and obtain

$$\frac{dV(N_t)}{dt} = -\gamma N(t) + (\alpha - \beta) F(N(t)) \leq [\alpha - (\gamma + \beta)] F(N(t)). \tag{4.3}$$

Define $u(r) = r$, $w(r) = (\gamma + \beta - \alpha) r$. Then we have

$$u(r) \to +\infty \quad \text{as} \quad r \to +\infty, \quad u(|\,\phi(0)\,|) \leq V(\phi) \tag{4.4}$$

and

$$\dot{V}(\phi) \leq -w(|\,\phi(0)\,|) \qquad \text{for } \forall \phi \in BC_+.$$

We conclude from Lemma 2.2 that the zero solution of (1.2) is globally asymptotically stable, and $BC_+$ is its attractive region. Thus we complete the proof.

Next, we investigate the case when (2.1) has a positive equilibrium $N(t) \equiv N^*$.

**Theorem 4.2.** *Assume that* (4.2) *holds. Let* $b \in (0, 1)$ *be chosen, and* $N_0 = bA$ *as in Lemma* 3.1 *or Lemma* 3.2. *Then any one of the following is the sufficient condition for the equilibrium* $N = N^*$ *of* (2.1) *to be globally asymptotically stable in* $BC_*$. *The attractive region is* $BC_*$.

1. $\alpha > \gamma > \alpha H$, $H = \dfrac{1 + (1-n) N_0^n}{(1 + N_0^n)^2}$ *for Eq.* (1.1);

2. $\alpha - \beta > \gamma > (\alpha - \beta) H$ *for Eq.* (1.2), $H$ *is the same as above.*

**Proof.** Let $y(t) = N(t) - N^*$. Then (1.2) becomes

$$\frac{dy(t)}{dt} = -\gamma y(t) - \beta F_1(y(t)) + \alpha \int_0^{\infty} K(s) F_1(y(t-s))\, ds, \quad t \geq 0, \tag{4.5}$$

where $F_1(y(t)) = F(y(t) + N^*) - F(N^*)$. We know that $y(t) \equiv 0$ is the solution of (4.5). Let

$$V(\phi) = |\phi(0)| + \alpha \int_{-\infty}^{0} \left( \int_{-s}^{\infty} K(u)\,du \right) |F_1(\phi(s))|\,ds \quad \text{for } \phi + N^* \in BC_*.$$

Then along with the solution of (4.5), we have

$$\frac{dV(y_t)}{dt} \le -\gamma|y(t)| - \beta\{\operatorname{sgn} y(t)\} F_1(y(t)) + \alpha|F_1(y(t))|. \tag{4.6}$$

Since $n \in (0, 1]$ and $F(u)$ is increasing for $u$, we have $y(t) F_1(y(t)) > 0$. Thus from (4.6),

$$\frac{dV(y_t)}{dt} \le -\gamma|y(t)| + (\alpha - \beta)|F_1(y(t))| = -\gamma|y(t)| + (\alpha - \beta) h(\xi(t))|y(t)|, \tag{4.7}$$

where $\xi(t) \ge 0$ lies between $N^*$ and $N^* + y(t)$, and

$$h(x) = \frac{1 + (1-n)x^n}{(1 + x^n)^2}, \quad x \ge 0,\ n \in (0, 1].$$

Noting that $h(x)$ has the following properties:

- $h(0) = 1$, $h(x) > 0$ for $x \in (0, \infty)$;
- $h'(x) < 0$, $h(x)$ is decreasing for $x \in (0, \infty)$, $\lim_{x \to +\infty} h(x) = 0$;

we get

$$0 \le h(x) \le h(N_0) \triangleq H \quad \text{for } x \ge N_0.$$

Since $N^* = \left( \frac{\alpha - \beta}{\gamma} - 1 \right)^{\frac{1}{n}} > A > N_0$, we derive from Lemma 3.1 that

$$\frac{dV(y_t)}{dt} \le -[\gamma - (\alpha - \beta)H]|y(t)|, \quad \text{as } N^* + y(t) \ge N_0. \tag{4.8}$$

Define $u(r) = r$, $w(r) = [\gamma - (\alpha - \beta)H]r$. Then we conclude from Lemma 2.2 that the equilibrium $y = 0$ of (4.5) is globally asymptotically stable.

Let $BC_0 = \{\phi \in BC_* : \phi(0) \geq N_0\}$. Then we have (4.8) if $N^* + y_t \in BC_0$. This implies that the positive equilibrium $N = N^*$ of (1.2) is globally asymptotically stable, and $BC_0$ is its attractive region. Note that (2.1) is autonomous and $N_t$ possesses the semi-group property. We conclude from Lemma 3.1 that for $\phi \in BC_*$, there is a $T > 0$ such that $N_T(\phi) \in BC_0$, hence $N_t(N_T(\phi)) = N_{t+T}(\phi) \in BC_0$ for $t \geq 0$. Thus (1.2) is globally asymptotically stable and the attractive region is $BC_*$.

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# A NOTE ON THE JACOBSON RADICAL

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Submitted by K. K. Azad

## Abstract

We prove a new spectral characterization of the Jacobson radical of a Banach algebra $A$ using the spectral radius and the group of invertible elements of $A$.

## Banach Algebras

We give a simple proof of a theorem of B. Aupetit [2, Theorem 1] characterizing the radical of a complex unital Banach algebra. It is well known from Zemanek's characterization of the radical that an element $a$ of a Banach algebra $A$ is in the Jacobson radical of $A$ if and only if $\rho(x + a) = \rho(x)$ for every $x \in A$, where $\rho$ denotes the spectral radius. First we prove that an element $a$ of a Banach algebra $A$ is in $\mathrm{Rad}(A)$ if and only if $\rho(xa) = 0$ for every $x \in \mathrm{Inv}(A)$.

**Lemma 1.1.** *Let $a$ be an element of a complex unital Banach algebra $A$. Then $a$ is in the Jacobson radical of $A$ if and only if $1 + ua$ is invertible for all $u \in \mathrm{Inv}(A)$.*

**Proof.** It is well known that if $a \in \mathrm{Rad}(A)$, then $1 + xa$ is invertible for all $x \in A$. To prove the converse, suppose that $1 + ua$ is invertible for

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all $u \in \mathrm{Inv}(A)$. Let $x \in A$ and choose $\alpha > 0$ such that $\alpha \| x \| < 1$. Since $\mathrm{Inv}(A)$ is a multiplicative group, we obtain

$$y = (\alpha + a)(\alpha x - 1)^{-1} \in \mathrm{Inv}(A)$$

and

$$y + a = (1 + ay^{-1})\, y \in \mathrm{Inv}(A).$$

Now, we have

$$(y + a)(\alpha x - 1) = \alpha + a\alpha x = \alpha(1 + ax) \in \mathrm{Inv}(A).$$

So $1 + ax \in \mathrm{Inv}(A)$ for all $x \in A$, hence $a \in \mathrm{Rad}(A)$.

**Corollary 1.2.** *Let $a$ be an element of a complex unital Banach algebra $A$. Then $a$ is in the Jacobson radical of $A$ if and only if $\rho(xa) = 0$ for all $x \in \mathrm{Inv}(A)$.*

Now, we can replace in Theorem 1 of [2] the strong condition for every $x \in A$ by the weaker one for every $x \in \mathrm{Inv}(A)$. We obtain the following theorem.

**Theorem 1.3.** *Let $a$ be an element of a complex unital Banach algebra $A$. Then $a$ is in the Jacobson radical of $A$ if and only if $\sup\{\rho(x + ta) : t \in C\} < \infty$ for every $x \in \mathrm{Inv}(A)$.*

**Proof.** If $a \in \mathrm{Rad}(A)$, it is well known by Zemanek's characterization of the radical that $\rho(x + ta) = \rho(x)$ for every $t \in C$ and every $x \in A$. So, $\sup\{\rho(x + ta) : t \in C\} < \infty$. Conversely, if the functions $t \mapsto \rho(x + ta)$ are bounded for every $x \in \mathrm{Inv}(A)$, then by Vesentini's theorem [1, p. 52] these functions are subharmonic and bounded, so by Liouville's theorem [1, p. 176], they are constant. Consequently, $\rho(x + ta) = \rho(x)$ for every $x \in \mathrm{Inv}(A)$ and $t \in C$. Then $1 - (x + ta)$ is invertible for every $t \in C$. The relation

$$1 - (x + ta) = (1 - x)[1 - (1 - x)^{-1} ta]$$

implies that $1/t \in \mathrm{Sp}((1 - x)^{-1} a)$ for every $t \in C$, so that $\rho((1 - x)^{-1} a)$

$< 1/t$ for every $t \in C$, i.e., $\rho((1-x)^{-1}a) = 0$. Now, for every $x \in \mathrm{Inv}(A)$, we have the relation

$$x^{-1} = (1-x)^{-1}u$$

with $u = (1 + (1-x)^{-1}x)^{-1} \in \mathrm{Inv}(A)$. So, we obtain the following

$$\rho((1-x)^{-1}a) = 0 \Leftrightarrow \rho(x^{-1}u^{-1}a) = 0$$

which by the preceding corollary implies that $u^{-1}a \in \mathrm{Rad}(A)$ and then $a \in \mathrm{Rad}(A)$ since $\mathrm{Rad}(A)$ is an ideal.

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# SPECIAL SUBSET IN *BCH*-ALGEBRAS

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## Abstract

We introduce the notion of *BCH*-algebras with condition (S), and show that a *BCH*-algebra $X$ with the additional identity $(x * y) * z = x * (y * (0 * z))$ for all $x, y, z \in X$ is a *BCH*-algebra with condition (S). We further discuss the structure of $A(x, y)$ and give a characterization of closed ideals in *BCH*-algebras.

## 1. Introduction

In 1966, Imai and Iséki [6] and Iséki [7] introduced two classes of abstract algebras: *BCK*-algebras and *BCI*-algebras. It is known that the class of *BCK*-algebras is a proper subclass of the class of *BCI*-algebras. In 1983, Hu and Li [5] introduced a wide class of abstract algebras: *BCH*-algebras. They have shown that the class of *BCI*-algebras is a proper subclass of the class of *BCH*-algebras. They have studied some properties of these algebras. Certain other properties have been studied by Ahmad [1], Chaudhry [2], Chaudhry and Fakhar-ud-din [3], Dudek and Thomys [4]. In 1977, Iséki introduced *BCK*-algebra with condition (S) in [8].

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Therefore, many others contributed to this subject. In this paper, we introduce the notion of *BCH*-algebras with condition (S), and show that a *BCH*-algebra $X$ with the additional identity $(x * y) * z = x * (y * (0 * z))$ for all $x, y, z \in X$ is a *BCH*-algebra with condition (S). Next, we discuss the structure of $A(x, y)$ and give a characterization of closed ideals in *BCH*-algebras.

## 2. Preliminaries

A *BCH-algebra* is a non-empty set $X$ with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(1) $x * x = 0$,

(2) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(3) $(x * y) * z = (x * z) * y$

for all $x, y, z$ in $X$. If $x * y = 0$ in $X$, then we say that $y$ is *greater* than $x$.

In any *BCH*-algebra $X$, the following hold:

(4) $(x * (x * y)) * y = 0$,

(5) $x * 0 = 0$ implies $x = 0$,

(6) $0 * (x * y) = (0 * x) * (0 * y)$,

(7) $x * 0 = x$.

In what follows, $X$ would mean a *BCH*-algebra unless otherwise specified.

A non-empty subset $S$ of $X$ is called a *subalgebra* if $x, y \in S$ implies $x * y \in S$. A non-empty subset $A$ of $X$ is called an *ideal* if $0 \in A$ and if $x * y, y \in A$ imply that $x \in A$. A non-empty subset $A$ of $X$ is called a *closed ideal* if

(i) $0 * x \in A$ for all $x \in A$ and

(ii) $x * y, y \in A$ imply that $x \in A$.

We know that every closed ideal in $X$ is a subalgebra but converse is not true.

## 3. Main Results

We introduce the notion of *BCH*-algebras with condition (S) and investigate its properties.

**Definition 3.1.** A *BCH*-algebra $X$ is called a *BCH-algebra with condition* (*S*) if for any $x,\ y \in X$, the set

$$A(x,\ y) := \{z \in X \mid (z * x) * y = 0\}$$

has the greatest element. This only greatest element is denoted by $x \circ y$.

From the definition it is easy to see that $x \circ 0 = 0 \circ x = x$ because $x \in A(0,\ x)$ and $0 \circ x \in A(0,\ x)$.

**Example 3.2.** Let $X = \{0,\ 1,\ 2,\ 3,\ 4\}$ be a set with Cayley table as follows:

| * | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 4 |
| 1 | 1 | 0 | 0 | 1 | 4 |
| 2 | 2 | 2 | 0 | 0 | 4 |
| 3 | 3 | 3 | 3 | 0 | 4 |
| 4 | 4 | 4 | 4 | 4 | 0 |

Then $(X;\ *,\ 0)$ is a *BCH*-algebra [2], but it is not with condition (S), since $A(1,\ 3) = \{0,\ 1,\ 2,\ 3\}$ has no any greatest element.

**Example 3.3.** Let $X = \{0,\ 1,\ 2,\ 3\}$ be a set with Cayley table as follows:

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 3 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 3 | 0 | 0 |

Then $(X; *, 0)$ is a $BCH$-algebra [10] with condition (S) and we can find the following $\circ$-table.

| $\circ$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 1 | 2 | 2 |
| 2 | 2 | 2 | 1 | 1 |
| 3 | 3 | 2 | 1 | 0 |

**Theorem 3.4.** *Suppose $x, y$ are any fixed elements of $X$. If $x \circ y$ exists, then so $y \circ x$, and $x \circ y = y \circ x$.*

**Proof.** Since $(z * x) * y = 0$ is equivalent to $(z * y) * x = 0$, $A(x, y) = A(y, x)$.

**Theorem 3.5.** *A BCH-algebra X with the additional identity*

(8) $(x * y) * z = x * (y * (0 * z))$ *for all* $x, y, z \in X$

*is a BCH-algebra with condition (S).*

**Proof.** For any $x, y \in X$, we show that $x \circ y = x * (0 * y)$. By (8), we have $((x * (0 * y)) * x) * y = (x * (0 * y)) * (x * (0 * y)) = 0$. This implies $x * (0 * y) \in A(x, y)$. On the other hand, let $u \in A(x, y)$. Then by (8), we get $u * (x * (0 * y)) = (u * x) * y = 0$. Thus we have shown $x \circ y = x * (0 * y)$. Therefore, $X$ is a $BCH$-algebra with condition (S).

**Theorem 3.6.** *If there is a binary operation $\vee$ on a BCH-algebra X such that for all $x, y, z \in X$,*

(9) $(x * y) * z = x * (y \vee z)$,

*then X is a BCH-algebra with condition (S) and $\vee$ is exactly the operation $\circ$.*

**Proof.** By (9), we obtain $((x \vee y) * x) * y = (x \vee y) * (x \vee y) = 0$. On the other hand, if $(z * x) * y = 0$, then $z * (x \vee y) = (z * x) * y = 0$. Hence $x \vee y$ is the greatest element of $A(x, y)$. Therefore, $X$ is with condition (S).

Now, we discuss the structure of $A(x, y)$.

**Theorem 3.7.** *If $I$ is an ideal of $X$, then* $I = \bigcup_{x,\, y \in I} A(x, y)$.

**Proof.** Let $I$ be an ideal of $X$. If $z \in I$, then since $(z * 0) * z = (z * z) * 0 = 0 * 0 = 0$, we have $z \in A(0, z)$. Hence $I \subseteq \bigcup_{z \in I} A(0, z) \subseteq \bigcup_{x,\, y \in I} A(x, y)$. Let $z \in \bigcup_{x,\, y \in I} A(x, y)$. Then there exist $a, b \in I$ such that $z \in A(a, b)$, so that $(z * a) * b = 0$. Since $I$ is an ideal, it follows that $z \in I$. Thus $\bigcup_{x,\, y \in I} A(x, y) \subseteq I$, and consequently $I = \bigcup_{x,\, y \in I} A(x, y)$.

**Corollary 3.8.** *If $I$ is an ideal of $X$, then* $I = \bigcup_{x \in I} A(0, x)$.

**Proof.** By Theorem 3.7, we have that $\bigcup_{x \in I} A(0, x) \subseteq \bigcup_{x,\, y \in I} A(x, y) = I$. If $x \in I$, then $x \in A(0, x)$ because $(x * 0) * x = 0$. Hence $I \subseteq \bigcup_{x \in I} A(0, x)$.

**Theorem 3.9.** *Let $I$ be a subset of $X$ such that $0 \in I$ and $I = \bigcup_{x,\, y \in I} A(x, y)$. Then $I$ is an ideal of $X$.*

**Proof.** Let $x * y,\ y \in I = \bigcup_{x,\, y \in I} A(x, y)$. It follows from (4) that $x \in A(x * y, y) \subseteq I$. Hence $I$ is an ideal of $X$.

Combining Theorems 3.7 and 3.9, we have the following corollary.

**Corollary 3.10.** *Let $I$ be a subset of $X$ containing 0. Then $I$ is an ideal of $X$ if and only if* $I = \bigcup_{x,\, y \in I} A(x, y)$.

Now we give a characterization of closed ideals.

**Theorem 3.11.** *Let $I$ be a subset of $X$. Then $I$ is a closed ideal of $X$ if and only if it satisfies*

(i) $0 \in I$,

(ii) $x * z \in I$, $y * z \in I$ *and* $z \in I$ *imply* $x * y \in I$.

**Proof.** Let $I$ be a closed ideal of $X$. Clearly $0 \in I$. Assume that $x * z$, $y * z$, $z \in I$. Since $I$ is an ideal, $x, y \in I$, which implies that $x * y \in I$ because $I$ is a closed ideal and hence a subalgebra.

Conversely, assume that $I$ satisfies (i) and (ii). Let $x * y$, $y \in I$. Since $0 * 0$, $y * 0$, $0 \in I$, by (ii) we have $0 * y \in I$. From (ii) again it follows that $x = x * 0 \in I$, so that $I$ is an ideal of $X$. Now suppose $x \in I$. Noticing that $0 * 0$, $x * 0$, $0 \in I$; $0 * x \in I$ follows from (ii).

**Theorem 3.12.** *Let I be an ideal of X. The set*

$$B := \{x \in I \mid 0 * x \in I\}$$

*is the greatest closed ideal of X which is contained in I.*

**Proof.** First we show that $B$ is an ideal of $X$. Clearly $0 \in B$. For any $x, y \in X$, if $x * y$, $y \in B$, then $0 * y \in I$ and $(0 * x) * (0 * y) = 0 * (x * y) \in I$. Since $I$ is an ideal of $X$, it follows that $0 * x \in I$. Moreover, since $B \subseteq I$, $x * y$, $y \in B \subseteq I$ implies $x \in I$. Hence $x \in B$, and so $B$ is an ideal of $X$. If $x \in B$, then $0 * x \in I$. Since $(0 * (0 * x)) * x = 0$, it follows that $0 * (0 * x) \in I$. Hence $0 * x \in B$, which proves that $B$ is closed. Now assume that $A$ is a closed ideal of $X$ which is contained in $I$. Let $x \in A$. Then $0 * x \in A$. Since $A$ is contained in $I$, $x$, $0 * x \in I$, and so $x \in B$. Thus $A \subseteq B$. Therefore, $B$ is the greatest closed ideal which is contained in $I$.

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# CERTAIN RESULTS INVOLVING $q$-SERIES AND CONTINUED FRACTIONS

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Submitted by K. K. Azad

## Abstract

In this paper we establish basic analogue of Entries 25 and 33 of Chapter XII of Ramanujan's Second Notebook.

## 1. Introduction

Nearly twelve years back, Singh [5] established the following result:

$$\frac{{}_3\Phi_2\left[\begin{matrix} a,\ b,\ c;\ \dfrac{de}{abc} \\ d,\ e \end{matrix}\right]}{{}_3\Phi_2\left[\begin{matrix} aq,\ b,\ c;\ \dfrac{de}{abc} \\ dq,\ e \end{matrix}\right]}$$

$$= 1 - \frac{\dfrac{de}{abc}(a-d)(1-b)(1-c)/(1-e)(1-d)(1-dq)}{\left(\dfrac{1-e/aq}{1-e}\right)+}$$

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$$\cfrac{\frac{e}{aq}(1-aq)\left(1-\frac{dq}{b}\right)\left(1-\frac{dq}{c}\right)\Big/(1-e)(1-dq)(1-dq^2)}{1-\cfrac{\frac{deq}{abc}(a-dq)(1-bq)(1-cq)/(1-eq)(1-dq^2)(1-dq^3)}{\left(\frac{1-e/aq}{1-eq}\right)+\cfrac{\frac{e}{aq}(1-aq^2)\left(1-\frac{d}{b}q^2\right)\left(1-\frac{d}{c}q^2\right)\Big/(1-eq)(1-dq^3)(1-dq^4)}{1-\cfrac{\frac{deq^2}{abc}(a-dq^2)(1-bq^2)(1-cq^2)/(1-eq^2)(1-dq^4)(1-dq^5)}{\left(\frac{1-e/aq}{1-eq^2}\right)+}}}}. \quad (1)$$

We shall make use of (1.1) and the following known transformation in order to establish basic analogues of Entries 33 and 25 of Chapter XII of Ramanujan's Second Notebook [4].

$$\begin{aligned}
&{}_3\Phi_2\left[\begin{matrix} a,\ b,\ c;\ \frac{de}{abc} \\ d,\ e \end{matrix}\right] \\
&= \frac{(e/b,\ e/c,\ cq/a,\ q/d;\ q)_\infty}{(e,\ cq/d,\ q/a,\ e/bc;\ q)_\infty}\ {}_3\Phi_2\left[\begin{matrix} c,\ d/a,\ cq/e;\ \frac{bq}{d} \\ cq/a,\ bcq/e \end{matrix}\right] \\
&- \frac{(q/d,\ eq/d,\ b,\ c,\ d/a,\ de/bcq,\ bcq^2/de;\ q)_\infty}{(d/q,\ e,\ bq/d,\ cq/d,\ q/a,\ e/bc,\ bcq/e;\ q)_\infty} \\
&\times\ {}_3\Phi_2\left[\begin{matrix} aq/d,\ bq/d,\ cq/d;\ \frac{de}{abc} \\ q^2/d,\ eq/d \end{matrix}\right]. \qquad [3;\ \text{App. III (III33)}] \quad (2)
\end{aligned}$$

## 2. Definitions and Notations

The basic hypergeometric series is defined as:

$$
{}_A\Phi_B\left[\begin{matrix}(a);\ z\\ (b)\end{matrix}\right]=\sum_{r=0}^{\infty}\frac{[(a);\ q]_r z^r}{[(b);\ q]_r(q;\ q)_r},\quad (|\,z\,|<1,\ |\,q\,|<1), \tag{3}
$$

where $(a;\ q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$; $(a;\ q)_0 = 1$ and $(a)$ stands for the sequence of $A$-parameters of the form $a_1,\ a_2,\ \dots,\ a_A$. The other notations appearing in this paper carry their usual meaning.

3. In this section, we shall establish our main results.

Taking $a = 1$, $d = q$ in (1.1) we find by simple transformation of continued fraction the following result:

$$
{}_3\Phi_2\left[\begin{matrix}q,\ b,\ c;\ eq/bc\\ q^2,\ e\end{matrix}\right]=\frac{1}{1-}\;\frac{\frac{eq}{bc}(1-q)(1-b)(1-c)\Big/(1-q)(1-q^2)}{(1-e/q)+}
$$

$$
\frac{\frac{e}{q}(1-q)(1-q^2/b)(1-q^2/c)\Big/(1-q^2)(1-q^3)}{1-}
$$

$$
\frac{\frac{eq^2}{bc}(1-q^2)(1-bq)(1-cq)\Big/(1-q^3)(1-q^4)}{(1-e/q)+}
$$

$$
\frac{\frac{e}{q}(1-q^2)(1-q^3/b)(1-q^3/c)\Big/(1-q^4)(1-q^5)}{1-}
$$

$$
\frac{\frac{eq^3}{bc}(1-q^3)(1-bq^2)(1-cq^2)\Big/(1-q^5)(1-q^6)}{(1-e/q)+\cdots}. \tag{4}
$$

Again, taking $c = q$ and $e = q^2$ in (1.2), we obtain

$$
{}_3\Phi_2\left[\begin{matrix}q,\ a,\ b;\ dq/ab\\ q^2,\ d\end{matrix}\right]
$$

$$
=\frac{(1-q)(1-q/d)}{(1-q/a)(1-q/b)}-\frac{(1-q/d)(1-q)(d/a,\ d/b;q)_\infty(q^2/a,\ q^2/b;\ q)_\infty}{(d/q,\ q/a,\ q/b;\ q)_\infty(dq/ab;\ q)_\infty}
$$

$$= \frac{(1-q)(1-q/d)}{(1-q/a)(1-q/b)} - \frac{(1-q)(1-q/d)}{(1-q/a)(1-q/b)} \frac{(d/a,\, d/b;\, q)_\infty}{(d/q,\, dq/ab;\, q)_\infty}. \tag{5}$$

Putting $c$ for $a$ and $e$ for $d$ in (3.5), we have

$$ {}_3\Phi_2\left[\begin{matrix} q,\, b,\, c;\, eq/bc \\ q^2,\, e \end{matrix}\right] = \frac{(1-q)(1-q/e)}{(1-q/b)(1-q/c)}\left[1 - \frac{(e/b,\, e/c;\, q)_\infty}{(e/q,\, eq/bc;\, q)_\infty}\right]. \tag{6}$$

Making use of (3.6) in (3.4), we get

$$\frac{(1-q)(1-q/e)}{(1-q/b)(1-q/c)}\left[1 - \frac{(e/b,\, e/c;\, q)_\infty}{(e/q,\, eq/bc;\, q)_\infty}\right]$$

$$= \frac{1}{1-} \frac{\frac{eq}{bc}(1-q)(1-b)(1-c)\Big/(1-q)(1-q^2)}{(1-e/q)+}$$

$$\frac{\frac{e}{q}(1-q)(1-q^2/b)(1-q^2/c)\Big/(1-q^2)(1-q^3)}{1-}$$

$$\frac{\frac{eq^2}{bc}(1-q^2)(1-bq)(1-cq)\Big/(1-q^3)(1-q^4)}{(1-e/q)+}$$

$$\frac{\frac{e}{q}(1-q^2)(1-q^3/b)(1-q^3/c)\Big/(1-q^4)(1-q^5)}{1-}$$

$$\frac{\frac{eq^3}{bc}(1-q^3)(1-bq^2)(1-cq^2)\Big/(1-q^5)(1-q^6)}{(1-e/q)+\cdots}. \tag{7}$$

Replacing $e$, $b$, $c$ by $q^{(x+m+n+3)/2}$, $q^{m+1}$ and $q^{n+1}$, respectively, we get

$$\frac{(1-q)\left(1-q^{-\left(\frac{x+m+n+1}{2}\right)}\right)}{(1-q^{-m})(1-q^{-n})}\left[1 - \frac{\Gamma_q\left(\frac{x+m+n+1}{2}\right)\Gamma_q\left(\frac{x-m-n+1}{2}\right)}{\Gamma_q\left(\frac{x-m+n+1}{2}\right)\Gamma_q\left(\frac{x+m-n+1}{2}\right)}\right]$$

$$= \frac{1}{1-}\,\frac{q^{\left(\frac{x-m-n+1}{2}\right)}(1-q)(1-q^{1+m})(1-q^{1+n})\Big/(1-q)(1-q^2)}{\left(1-q^{\frac{x+m+n+1}{2}}\right)+}$$

$$\frac{q^{\left(\frac{x+m+n+1}{2}\right)}(1-q)(1-q^{1-m})(1-q^{1-n})\Big/(1-q^2)(1-q^3)}{1-}$$

$$\frac{q^{\left(\frac{x-m-n+3}{2}\right)}(1-q^2)(1-q^{2+m})(1-q^{2+n})\Big/(1-q^3)(1-q^4)}{\left(1-q^{\frac{x+m+n+1}{2}}\right)+}$$

$$\frac{q^{\left(\frac{x+m+n+1}{2}\right)}(1-q^2)(1-q^{2-m})(1-q^{2-n})\Big/(1-q^4)(1-q^5)}{1-}$$

$$\frac{q^{\left(\frac{x-m-n+5}{2}\right)}(1-q^3)(1-q^{3+m})(1-q^{3+n})\Big/(1-q^5)(1-q^6)}{\left(1-q^{\frac{x+m+n+1}{2}}\right)+\cdots}. \tag{8}$$

As $q \to 1$, (3.8) yields Entry 33 after some simple manipulations.

Again, taking first $a = 1$, and then $d = 1$ in (1.1), we get

$$\frac{\left(\frac{e}{b}, \frac{e}{c}; q\right)_\infty}{(e, e/bc; q)_\infty} = \frac{1}{1-}\,\frac{\frac{e}{bc}(1-b)(1-c)\Big/(1-q)}{(1-e/q)+}$$

$$\frac{\frac{e}{q}(1-q)(1-q/b)(1-q/c)\Big/(1-q)(1-q^2)}{1-}$$

$$\frac{\frac{eq}{bc}(1-q)(1-bq)(1-cq)\Big/(1-q^2)(1-q^3)}{(1-e/q)+}$$

$$\cfrac{\frac{e}{q}(1-q^2)(1-q^2/b)(1-q^2/c)\Big/(1-q^3)(1-q^4)}{1- \cfrac{\frac{eq^2}{bc}(1-q^2)(1-bq^2)(1-cq^2)\Big/(1-q^4)(1-q^5)}{(1-e/q)+\cdots}}. \quad (9)$$

By simple transformation of the continued fraction, (3.9) can be written as,

$$\frac{(e/b,\, e/c;\, q)_\infty}{(e/q,\, e/bc;\, q)_\infty} = \cfrac{1}{(1-e/q)- \cfrac{\frac{e}{bc}(1-b)(1-c)\Big/(1-q)}{1+ \cfrac{\frac{e}{q}(1-q/b)(1-q/c)\Big/(1-q^2)}{(1-e/q)- \cfrac{\frac{eq}{bc}(1-q)(1-bq)(1-cq)\Big/(1-q^2)(1-q^3)}{1+ \cfrac{\frac{e}{q}(1-q^2)(1-q^2/b)(1-q^2/c)\Big/(1-q^3)(1-q^4)}{(1-e/q)- \cfrac{\frac{eq^2}{bc}(1-q^2)(1-bq^2)(1-cq^2)\Big/(1-q^4)(1-q^5)}{1+\cdots}}}}}}. \quad (10)$$

Replacing $e$, $b$, $c$ by $q^{(x+n+5)/4}$, $q^{(n+1)/2}$, $q^{1/2}$, respectively, in (3.10), we get

$$\frac{\Gamma_q\left(\frac{x+n+1}{4}\right)\Gamma_q\left(\frac{x-n+1}{4}\right)}{\Gamma_q\left(\frac{x+n+3}{4}\right)\Gamma_q\left(\frac{x-n+3}{4}\right)}$$

$$= \cfrac{(1-q)}{(1-q^{(x+n+1)/4})- \cfrac{q^{(x-n+1)/4}(1-q^{(n+1)/2})(1-q^{1/2})\Big/(1-q)}{1+}}$$

$$\frac{q^{(x+n+1)/4}(1-q^{(1-n)/2})(1-q^{1/2})/(1-q^2)}{(1-q^{(x+n+1)/4})-}$$

$$\frac{q^{(x-n+5)/4}(1-q)(1-q^{(n+3)/2})(1-q^{3/2})/(1-q^2)(1-q^3)}{1+}$$

$$\frac{q^{(x+n+1)/4}(1-q^2)(1-q^{(3-n)/2})(1-q^{3/2})/(1-q^3)(1-q^4)}{(1-q^{(x+n+1)/4})-\cdots}. \qquad (11)$$

As $q \to 1$, (3.11) yields Entry 25 after some simple manipulations.

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# ON DENJOY-LUSIN'S THEOREM

Dedicated to Professor Kôzô Yabuta on his sixtieth birthday

**KAORU YONEDA**

( Received April 24, 2001 )

Submitted by K. K. Azad

## Abstract

Let $E$ be an $M$-set in the narrow sense. If $\rho_n \geq 0$ and if

$$\sum_{n=0}^{\infty} \rho_n |\cos(nx + \theta_n)|^p < \infty \quad \text{in } E$$

for some natural number $p$ and $0 \leq \theta_n < 2\pi$, then

$$\sum_{n=0}^{\infty} \rho_n < \infty.$$

## 1. Introduction

The classical Denjoy-Lusin's theorem says that if $E$ is a measure positive subset of $[-\pi, \pi]$ and if

$$\sum_{n=0}^{\infty} \rho_n |\cos(nx + \theta_n)| < \infty \quad \text{in } E$$

for $\rho_n \geq 0$ (throughout this paper we assume that $\rho_n \geq 0$ for $n = 0, 1, 2, \ldots$) and $0 \leq \theta_n < 2\pi$, then

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$$\sum_{n=0}^{\infty} \rho_n < \infty \tag{1.1}$$

(see Bary [1], p. 173 or Zygmund [3], Chapter 6).

When a subset of $[-\pi, \pi]$, which does not need to be of measure positive, has above property, then it is called an *A.C. set*.

If $\frac{x}{\pi}$ is a rational number, then

$$\sum_{n=1}^{\infty} |\sin n!\, x| < \infty$$

but this series does not satisfy (1.1), since $\rho_{n!} = 1$ for all $n$. Thus there exists a denumerable and dense set which is not an *A.C.* set.

Afterward, the Denjoy-Lusin's theorem was improved such as a set of second category or a basis in an *A.C.* set. A set of positive measure is of the second category (see Bary [1], p. 532). The Cantor set, which is of the first category, is a basis. Hence they are *A.C.* sets.

In this paper, we shall show that an *M*-set in the narrow sense is an *A.C.* set, moreover, a generalized *A.C.* set. The concept of *M*-set in the narrow sense, which is of the first category, was introduced by Men'shov [2].

## 2. Main Theorem

A subset $E$ of $[-\pi, \pi]$ is called an *M-set in the narrow sense for trigonometric series* if there exists a non-negative measure $\mu$ such that

$\mu(S) \geq 0$ for all measurable set $S$;

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \mu(dx) = \frac{1}{\pi}\int_{E} \mu(dx) = \frac{1}{\pi}\mu(E) = 1;$$

$\mu(S) = 0$ for all measurable set $S$ such that $S \cap E = \emptyset$;

$$\lim_{n\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx\mu(dx) = \lim_{n\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx\mu(dx) = 0.$$

A subset $E$ of $[-\pi, \pi]$ is called a *pA.C. set* for $p = 1, 2, \ldots,$ if

$$\sum_{n=0}^{\infty} \rho_n |\cos(nx + \theta_n)|^p < \infty \quad \text{in } E$$

includes (1.1).

**Theorem 2.1.** *An M-set in the narrow sense is a pA.C. set for* $p = 1, 2, \ldots$ .

**Proof.** First we shall prove the theorem when $p$ is an integer which has the form $p = 4m$. Put

$$T(x) = \sum_{n=0}^{\infty} \rho_n |\cos(nx + \theta_n)|^{4m},$$

then $T(x)$ is finite in $E$. Put

$$E_N = \{x \in E : T(x) < N\}$$

for $N = 1, 2, \ldots$ .

Since $E$ is an $M$-set in the narrow sense, there exists an integer $N$ such that

$$\frac{1}{\pi} \int_{E_N} \mu(dx) = \frac{1}{\pi} \mu(E_N) > \frac{1}{2}.$$

$T(x)$ is less than $N$ in $E_N$. Thus, by the Lebesgue's convergence theorem, we have

$$\frac{1}{\pi} \int_{E_N} T(x)\, \mu(dx) = \frac{1}{\pi} \sum_{n=0}^{\infty} \rho_n \int_{E_N} |\cos(nx + \theta_n)|^{4m} \mu(dx)$$

$$\leq \frac{N}{\pi} \int_{E_N} \mu(dx) = \frac{N}{\pi} \mu(E_N).$$

Put

$$I_n \equiv \frac{1}{\pi} \int_{E_N} |\cos(nx + \theta_n)|^{4m} \mu(dx)$$

$$= \frac{1}{\pi} \int_{E_N} \left\{ \frac{1 + \cos 2(nx + \theta_n)}{2} \right\}^{2m} \mu(dx)$$

$$= \frac{1}{4^m \pi} \int_{E_N} \sum_{j=0}^{2m} \binom{2m}{j} \cos^j (2nx + 2\theta_n) \, \mu(dx).$$

The following two equalities are well known:

$$\cos^{2j} u = \frac{2}{4^j} \left\{ \sum_{r=0}^{j-1} \binom{2j}{r} \cos(2j - 2r) u + \frac{1}{2} \binom{2j}{j} \right\};$$

$$\cos^{2j+1} u = \frac{1}{4^j} \left\{ \sum_{r=0}^{j} \binom{2j+1}{r} \cos(2j - 2r + 1) u \right\}.$$

Then we have

$$I_n = \frac{1}{4^m \pi} \sum_{j=0}^{2m} \binom{2m}{j} \int_{E_N} \cos^j (2(nx + \theta_n)) \, \mu(dx)$$

$$= \frac{1}{4^m \pi} \binom{2m}{0} \mu(E_N) + \frac{1}{4^m \pi} \sum_{j=1}^{2m} \binom{2m}{j} \int_{E_N} \cos^j (2(nx + \theta_n)) \, \mu(dx)$$

$$= \frac{1}{4^m \pi} \mu(E_N) + \frac{1}{4^m \pi} \sum_{j=1}^{m} \binom{2m}{2j} \int_{E_N} \cos^{2j} (2(nx + \theta_n)) \, \mu(dx)$$

$$+ \frac{1}{4^m \pi} \sum_{j=0}^{m-1} \binom{2m}{2j+1} \int_{E_N} \cos^{2j+1} (2(nx + \theta_n)) \, \mu(dx)$$

$$= \frac{1}{4^m \pi} \mu(E_N) + \frac{1}{4^m \pi} \sum_{j=1}^{m} \binom{2m}{2j} \int_{E_N} \frac{2}{4^j} \sum_{r=0}^{j-1} \binom{2j}{r} \cos(2j - 2r) (2(nx + \theta_n))$$

$$+ \frac{1}{4^j} \binom{2j}{j} \mu(dx)$$

$$+\frac{1}{4^m\pi}\sum_{j=0}^{m-1}\binom{2m}{2j+1}\int_{E_N}\frac{1}{4^j}\sum_{r=0}^{k}\binom{2j+1}{r}\cos(2j-2r+1)(2(nx+\theta_n))\mu(dx)$$

$$=\frac{1}{4^m\pi}\mu(E_N)+\frac{1}{4^m\pi}\sum_{j=1}^{m}\binom{2m}{2j}\frac{2}{4^j}\sum_{r=0}^{j-1}\binom{2j}{r}\int_{E_N}\cos(4j-4r)(nx+\theta_n)\mu(dx)$$

$$+\frac{1}{4^m\pi}\sum_{j=1}^{m}\binom{2m}{2j}\frac{1}{4^j}\binom{2j}{j}\mu(E_N)$$

$$+\frac{1}{4^m\pi}\sum_{j=0}^{m-1}\binom{2m}{2j+1}\frac{1}{4^j}\sum_{r=0}^{j}\binom{2j+1}{r}\int_{E_N}\cos(4j-4r+2)(nx+\theta_n)\mu(dx)$$

$$=\frac{1}{4^m\pi}\left\{1+\sum_{j=1}^{m}\binom{2m}{2j}\frac{1}{4^j}\binom{2j}{j}\right\}\mu(E_N)$$

$$+\frac{1}{4^m\pi}\sum_{j=1}^{m}\binom{2m}{2j}\frac{2}{4^j}\sum_{r=0}^{j-1}\binom{2j}{r}\int_{E_N}\cos(4j-4r)(nx+\theta_n)\mu(dx)$$

$$+\frac{1}{4^m\pi}\sum_{j=0}^{m-1}\binom{2m}{2j+1}\frac{1}{4^j}\sum_{r=0}^{j}\binom{2j+1}{r}\int_{E_N}\cos(4j-4r+2)(nx+\theta_n)\mu(dx).$$

Since for any $\varepsilon > 0$ there exist sufficiently large $N$ and $n$ such that

$$\left|\frac{1}{\pi}\int_{E_N}\cos nx\mu(dx)\right| = \left|\frac{1}{\pi}\int_{E}\cos nx\mu(dx) - \int_{E\setminus E_N}\cos nx\mu(dx)\right|$$

$$\le\left|\frac{1}{\pi}\int_{E}\cos nx\mu(dx)\right| + \left|\frac{1}{\pi}\int_{E\setminus E_N}\cos nx\mu(dx)\right| < \varepsilon+\frac{1}{\pi}\mu(E\setminus E_N) < 2\varepsilon$$

and similarly

$$\left|\frac{1}{\pi}\int_{E_N}\sin nx\mu(dx)\right| < 2\varepsilon,$$

we have

$$I_n \geq C\mu(E_N) - c\varepsilon$$

for all $n > n_0$ where $C$ and $c$ are some positive constants. Thus by the following inequality

$$\frac{1}{\pi} N\mu(E_N) \geq \frac{1}{\pi} \sum_{n=0}^{\infty} \rho_n I_n \geq \frac{1}{\pi} \sum_{n=n_0+1}^{\infty} \rho_n (C\mu(E_N) - c\varepsilon)$$

and by the fact $C\mu(E_N) - c\varepsilon > 0$, we have

$$\frac{1}{C\mu(E_N) - c\varepsilon} N\mu(E_N) \geq \sum_{n_0+1}^{\infty} \rho_n.$$

Consequently, we have (1.1). Since

$$\sum_{n=0}^{\infty} \rho_n |\cos(nx + \theta_n)|^{4m-3} \geq \sum_{n=0}^{\infty} \rho_n |\cos(nx + \theta_n)|^{4m-2}$$

$$\geq \cdots \geq \sum_{n=0}^{\infty} \rho_n |\cos(nx + \theta_n)|^{4m},$$

it is easy to prove the theorem for other cases.

**Corollary 2.2.** *An M-set in the narrow sense is an A.C. set.*

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# δ-SEMI-OPEN SETS AND ITS APPLICATIONS

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## Abstract

In this paper, we study topological properties of δ-semi-open sets which were defined in [J. Indian Acad. Math. 19 (1997), 59-67] and introduce several operators related to δ-semi-open sets.

## 1. Introduction and Preliminaries

Levine [5] defined semi-open sets which are weaker than open sets and their related semi-continuity in topological spaces. After Levine's semi-open set, mathematicians gave in several papers different and interesting new open sets as well as generalized open sets. In 1968, Veličko [12] introduced δ-open sets, which are stronger than open sets, in order to investigate the characterization of $H$-closed spaces and showed that $\tau_\delta$ (= the collection of all δ-open sets) is the topology in $X$ such that $\tau_\delta \subset \tau$ and $\tau_\delta$ equal with the semi-regularization topology $\tau_s$. In 1997, Park, Lee and Son [10] have introduced the notion of δ-semi-open sets which are stronger than semi-open sets but weaker than δ-open sets and investigated the relationships between several types of these open sets.

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Some operators using Levine's semi-open sets were introduced by Das [3] in 1971 were certain relationships among semi-interior, semi-closure and other notions.

The aim of this paper is to continue the study topological properties of $\delta$-semi-open sets which were defined by Park, Lee and Son [10]. Also, we introduce the notions of $\delta$-semi-limit-point, $\delta$-semi-derived set, $\delta$-semi-border, $\delta$-semi-frontier and $\delta$-semi-exterior of a set using the concept of $\delta$-semi-open set and obtain some of their elementary properties and study the properties of $\delta$-semi-closure and $\delta$-semi-interior which were defined in [10].

All spaces considered in this paper lack separation axioms unless explicitly stated. The topology of a space is denoted by $\tau$ and $(X, \tau)$ will be replaced by $X$ if there is no chance for confusion. For a subset $A \subset X$, the closure, the interior and the complement of $A$ in $X$ are denoted by $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$ and $X \setminus A$ respectively.

**Definition 1.1** ([3], [5]). Let $A$ be a subset of a space $X$.

(1) $A$ is said to be *semi-open set* if there exists an open set $U$ of $X$ such that $U \subset A \subset \mathrm{Cl}(U)$.

(2) $A$ is said to be *$\delta$-open set* if $A = \mathrm{Int}_\delta(A)$, i.e., a set is $\delta$-open if it is the union of regular open sets.

(3) $x \in X$ is said to be *semi-limit point* of $A$ if for each semi-open set $U$ containing $x$, $U \cap (A \setminus \{x\}) \neq \varnothing$.

(4) $\mathrm{sd}(A) = \{x \in X \mid x \text{ is semi-limit point of } A\}$ is called the *semi-derived set* of $A$.

(5) $x \in X$ is said to be *semi-interior point* of $A$ if there exists a semi-open set $U$ such that $x \in U \subset A$.

(6) $\mathrm{sb}(A) = A \setminus s\mathrm{Int}(A)$ is called the *semi-border* of $A$.

(7) $\mathrm{sf}(A) = s\mathrm{Cl}(A) \setminus s\mathrm{Int}(A)$ is called the *semi-frontier* of $A$.

(8) $\mathrm{se}(A) = s\mathrm{Int}(X \setminus A)$ is called the *semi-exterior* of $A$.

## 2. Properties of δ-semi-open Sets

A subset $A$ of a space $X$ is said to be a *δ-semi-open set* [10] if there exists a δ-open set $U$ of $X$ such that $U \subset A \subset \mathrm{Cl}(U)$. The collection of all δ-semi-open sets of $X$ is denoted by δ-SO($X$). The complement of a δ-semi-open set is called a *δ-semi-closed set*.

Every δ-open set is δ-semi-open and every δ-semi-open set is semi-open. The following Example 2.1 shows that the converses are not true and the concepts of open set and δ-semi-open set are independent of each other.

**Example 2.1.** Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then $\{c, d\}$ is δ-semi-open but neither δ-open nor open. Also, $\{a, c, d\}$ is open and hence semi-open but not δ-semi-open.

**Theorem 2.2.** *Any union (resp. intersection) of δ-semi-open (resp. δ-semi-closed) sets of $X$ is δ-semi-open (resp. δ-semi-closed).*

The intersection of two δ-semi-open sets may not be δ-semi-open as the following example shows. Thus the δ-semi-open sets of a space $X$ do not always form a topology.

**Example 2.3.** Consider the space $(X, \tau)$ given in Example 2.1. Then $\{c, d\}$ and $\{a, b, d\}$ are δ-semi-open but their intersection $\{d\}$ is not δ-semi-open.

**Lemma 2.4.** *The intersection of δ-open set and δ-semi-open set is a δ-semi-open.*

If in above lemma δ-open set is replaced by open set, then the result need not true, i.e., the intersection of open set and δ-semi-open set is not δ-semi-open.

**Example 2.5.** Consider the space given in Example 2.1. Then $\{a, b\}$ is δ-semi-open and $\{a, c\}$ is open but their intersection $\{a\}$ is not δ-semi-open.

**Lemma 2.6.** *$A$ is δ-semi-open if and only if* $\mathrm{Cl}(A) = \mathrm{Cl}(\delta\text{-Int}(A))$.

**Proof.** If $A$ is δ-semi-open, then by the definition, $A \subset \mathrm{Cl}(\delta\text{-}\mathrm{Int}(A))$ and hence $\mathrm{Cl}(A) \subset \mathrm{Cl}(\delta\text{-}\mathrm{Int}(A))$. Conversely, if $\mathrm{Cl}(A) = \mathrm{Cl}(\delta\text{-}\mathrm{Int}(A))$, then we have $\delta\text{-}\mathrm{Int}(A) \subset A \subset \mathrm{Cl}(A) = \mathrm{Cl}(\delta\text{-}\mathrm{Int}(A))$. Hence $A$ is δ-semi-open.

**Lemma 2.7.** *If $A$ is δ-semi-open and $A \neq \varnothing$, then $\delta\text{-}\mathrm{Int}(A) \neq \varnothing$.*

The δ-*semi-closure* [10] (resp. δ-*semi-interior* [10]) is the intersection (resp. union) of all δ-semi-closed (resp. δ-semi-open) sets of $X$ containing (resp. contained in) $A$ and is denoted by $\delta\text{-}s\mathrm{Cl}(A)$ (resp. $\delta\text{-}s\mathrm{Int}(A)$).

**Theorem 2.8.** *For a subset $A$ of $X$, the following statements are true:*

(1) $s\mathrm{Cl}(A) \subset \delta\text{-}s\mathrm{Cl}(A)$.

(2) *$A$ is δ-semi-closed in $X$ if and only if $A = \delta\text{-}s\mathrm{Cl}(A)$.*

(3) *If $A \subset B$, then $\delta\text{-}s\mathrm{Cl}(A) \subset \delta\text{-}s\mathrm{Cl}(B)$.*

(4) $\delta\text{-}s\mathrm{Cl}(A) \cup \delta\text{-}s\mathrm{Cl}(B) \subset \delta\text{-}s\mathrm{Cl}(A \cup B)$.

(5) $\delta\text{-}s\mathrm{Cl}(A) \cap \delta\text{-}s\mathrm{Cl}(B) \supset \delta\text{-}s\mathrm{Cl}(A \cap B)$.

(6) $\delta\text{-}s\mathrm{Cl}(\delta\text{-}s\mathrm{Cl}(A)) = \delta\text{-}s\mathrm{Cl}(A)$.

(7) *$x \in \delta\text{-}s\mathrm{Cl}(A)$ if and only if $A \cap V \neq \varnothing$ for each δ-semi-open set $V$ containing $x$.*

**Proof.** We prove only (6) and (7).

(7) Suppose that there exists a δ-semi-open set $V$ containing $x$ such that $A \cap V \neq \varnothing$. Then $A \subset X \setminus V$ and $X \setminus V$ is a δ-semi-closed set. Since $\delta\text{-}s\mathrm{Cl}(A) \subset X \setminus V$, $x \notin \delta\text{-}s\mathrm{Cl}(A)$. Conversely, suppose that $x \notin \delta\text{-}s\mathrm{Cl}(A)$. Put $V = X \setminus \delta\text{-}s\mathrm{Cl}(A)$. Then $V$ is δ-semi-open set containing $x$ and $A \cap V = \varnothing$. Hence $x \in \delta\text{-}s\mathrm{Cl}(A)$.

The following example shows that equality in Theorem 2.8(1) does not hold.

**Example 2.9.** Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$. Take $A = \{a, b\}$. Then $\delta\text{-}s\mathrm{Cl}(A) = X$ and $s\mathrm{Cl}(A) = \{a, b\}$ and hence $\delta\text{-}s\mathrm{Cl}(A) \neq s\mathrm{Cl}(A)$.

**Theorem 2.10.** *Let $\tau_\delta$ be the family of δ-open sets of X. Then the following statements are true:*

(1) $\tau_\delta \subset \delta\text{-SO}(X)$.

(2) *If A is a δ-semi-open set of X and $A \subset B \subset \mathrm{Cl}(A)$, then B is a δ-semi-open set of X.*

**Proof.** (1) Since every δ-open set is δ-semi-open, $\tau_\delta \subset \delta\text{-SO}(X)$.

(2) Since $A$ is a δ-semi-open set, there exists a δ-open set $U$ such that $U \subset A \subset \mathrm{Cl}(U)$. Then $U \subset B$. But $\mathrm{Cl}(A) \subset \mathrm{Cl}(U)$ and hence $B \subset \mathrm{Cl}(U)$. Thus $B$ is δ-semi-open.

**Theorem 2.11.** *If Y is a δ-open set of X and A is a δ-semi-open set of X, then $A \cap Y$ is a δ-semi-open set of Y.*

**Theorem 2.12.** *Let Y be an open subspace of X and $A \subset Y$. If A is a δ-semi-open set of X, then A is also a δ-semi-open set of Y.*

**Proof.** Since $A$ is δ-semi-open, there exists a δ-open set $U$ of $X$ such that $U \subset A \subset \mathrm{Cl}(U)$. Thus $U$ is δ-semi-open in $Y$ and $U = U \cap Y \subset A \cap Y \subset \mathrm{Cl}(U) \cap Y$, i.e., $U \subset A \subset \mathrm{Cl}_Y(U)$. Hence $A$ is δ-semi-open in $Y$.

If $Y$ is any subspace of $X$ and $A$ is δ-semi-open in $Y$, then $A$ may not be δ-semi-open in $X$. However, we have

**Theorem 2.13.** *Let Y be an open subspace of X. If A is δ-semi-open in Y and Y is δ-semi-open in X, then A is δ-semi-open in X.*

**Proof.** Since $A$ is δ-semi-open in $Y$, there exists a δ-open set $U_Y$ of $Y$ such that $U_Y \subset A \subset \mathrm{Cl}_Y(U_Y)$. Also, there exists a δ-open set $U$ of $X$ such that $U_Y = U \cap Y$. Then $U \cap Y$ is a δ-semi-open set from Lemma 2.4 and $U \cap Y \subset A \subset \mathrm{Cl}_Y(U \cap Y) \subset \mathrm{Cl}(U \cap Y)$. Hence, by Theorem 2.10, $A$ is δ-semi-open in $X$.

**Example 2.14.** Let $X = \{a, b, c, d\}$, $Y = \{b, c\}$ and $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then $\{b\}$ is δ-semi-open in $Y$ but not δ-semi-open in $X$.

**Lemma 2.15.** *If $A$ is a δ-open set of $X$, then $\mathrm{Cl}(A) \setminus A$ is nowhere dense in $X$.*

**Theorem 2.16.** *If $A$ is a δ-semi-open set of $X$, then $A = U \cup B$ where $U$ is δ-open in $X$ and $B$ is nowhere dense in $X$ such that $U \cap B = \varnothing$.*

**Proof.** Since $A$ is δ-semi-open in $X$, there exists a δ-open set $U$ of $X$ such that $U \subset A \subset \mathrm{Cl}(U)$. Let $A = U \cup (A \setminus U)$ and $B = A \setminus U$. Then $B \subset \mathrm{Cl}(A) \setminus U \subset \mathrm{Cl}(U) \setminus U$. Therefore $B \subset \mathrm{Cl}(U) \setminus U$ and by Lemmas 2.5 and 2.15, $B$ is nowhere dense in $X$. Hence $A = U \cup B$ where $U$ is δ-open in $X$, $B$ is nowhere dense in $X$ and $U \cap B = \varnothing$.

The converse of above theorem is false as is shown by the following example.

**Example 2.17.** Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$. Then $\{c\}$ is δ-open set, $\{d\}$ is nowhere dense in $(X, \tau)$ and $\{c\} \cap \{d\} = \varnothing$. But $\{c, d\}$ is not a δ-semi-open set.

**Definition 2.18.** Let $A$ be a subset of a space $X$. A point $x \in X$ is said to be a δ-*limit point* of $A$ if for each δ-open set $U$ containing $x$, $U \cap (A \setminus \{x\}) \neq \varnothing$. The set of all δ-limit points of $A$ is said to be a δ-*derived set* of $A$ and is denoted by δ-d$(A)$.

**Theorem 2.19.** *If $A = U \cup B$ where $U$ is a nonempty δ-open set, $A$ is a δ-connected set and the δ-derived set δ-d$(B)$ of $B$ is empty, then $A$ is a δ-semi-open set.*

**Proof.** It is sufficient to show that $B \subset \delta\text{-}\mathrm{Cl}(U)$. If not, then $B = B_1 \cup B_2$ where $B_1 \subset \delta\text{-}\mathrm{Cl}(U)$ and $B_2 \subset X \setminus \delta\text{-}\mathrm{Cl}(U)$. Therefore $A = (U \cup B_1) \cup B_2$ and $B_2 \neq \varnothing$ and $U \cup B_1 \neq \varnothing$. Also, $U \cup B_1 \subset \delta\text{-}\mathrm{Cl}(U)$ is closed set, and since $\delta\text{-d}(B_2) \subset \delta\text{-d}(B) = \varnothing$, $B_2$ is δ-closed set and $B_2 \cap \delta\text{-}\mathrm{Cl}(U) = \varnothing$. Thus, $U \cup B_1$ and $B_2$ constitute a δ-separation

for $A$. It is contradiction to the fact that $A$ is δ-connected. Hence $B \subset \delta\text{-Cl}(U)$.

The converse of above theorem is not true. In other words, it is not true that components of δ-semi-open sets are δ-semi-open. We can show this by the following example.

**Example 2.20.** Let $X$ be the set of real numbers with usual topology and $A = \{0\} \cup \left(\frac{1}{2}, 1\right) \cup \left(\frac{1}{4}, \frac{1}{2}\right) \cup \cdots \cup \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) \cup \cdots$. Then $A$ is δ-semi-open in $X$ and $\{0\}$ is a component of $A$, but $\{0\}$ is not δ-semi-open in $X$.

**Theorem 2.21.** *Let $f : X \to Y$ be a function.*

(1) *If $f$ is a δ-open continuous function and $A$ is δ-semi-open in $X$, then $f(A)$ is δ-semi-open in $Y$.*

(2) *If $f$ is a almost open δ-continuous function and $B$ is δ-semi-open in $Y$, then $f^{-1}(B)$ is δ-semi-open in $X$.*

**Proof.** (1) Since $A$ is δ-semi-open, by Theorem 2.16, $A = U \cup B$ where $U$ is δ-open and $B \subset \text{Cl}(U) \setminus U$. Then $f(U) \subset f(A) = f(U \cup B) \subset f(U) \cup f(\text{Cl}(U)) \subset f(U) \cup \text{Cl}(f(U)) = \text{Cl}(f(U))$. Since $f$ is δ-open, $f(U)$ is δ-open in $Y$ and $f(U) \subset f(A) \subset \text{Cl}(f(U))$. This shows that $f(A)$ is δ-semi-open in $Y$.

(2) Since $B$ is δ-semi-open, there exists a δ-open set $V$ of $Y$ such that $V \subset B \subset \text{Cl}(V)$. By assumption, $f^{-1}(V)$ is δ-open in $X$ and $f^{-1}(V) \subset f^{-1}(B) \subset f^{-1}(\text{Cl}(V)) \subset \text{Cl}(f^{-1}(V))$.

**Lemma 2.22.** *Let $\tau_\delta$ be the family of δ-open sets of a space $X$. Then $\tau_\delta = \delta\text{-Int}(\delta\text{-SO}(X))$.*

**Proof.** If $U \in \tau_\delta$, then $U \in \delta\text{-SO}(X)$ and since $U = \delta\text{-Int}(U)$, $U \in \delta\text{-Int}(\delta\text{-SO}(X))$. Conversely, let $U \in \delta\text{-Int}(\delta\text{-SO}(X))$. Then $U = \delta\text{-Int}(A)$ for some δ-semi-open set $A$ of $X$. Hence $U \in \tau_\delta$.

**Theorem 2.23.** *Let* $(X, \tau)$ *and* $(X, \tau^*)$ *be topological spaces. If* $\delta\text{-SO}(X, \tau) \subset \delta\text{-SO}(X, \tau^*)$, *then* $\tau_\delta \subset \tau^*_\delta$.

**Corollary 2.24.** *Let* $(X, \tau)$ *and* $(X, \tau^*)$ *be topological spaces. If* $\delta\text{-SO}(X, \tau) = \delta\text{-SO}(X, \tau^*)$, *then* $\tau_\delta = \tau^*_\delta$.

**Theorem 2.25.** *If* $A_i$ *is a* $\delta$*-semi-open set of* $X_i$ $(i = 1, 2)$, *then* $A_1 \times A_2$ *is* $\delta$*-semi-open in* $X_1 \times X_2$.

**Proof.** For $i = 1, 2$, there exists a $\delta$-open set $U_i$ such that $U_i \subset A_i \subset \text{Cl}_{X_i}(U_i)$. Then, by Lemma 2.6, we have $U_1 \times U_2$ is $\delta$-open in $X$ and

$$U_1 \times U_2 \subset A_1 \times A_2 \subset \text{Cl}_{X_1}(U_1) \times \text{Cl}_{X_2}(U_2) = \text{Cl}_{X_1 \times X_2}(U_1 \times U_2).$$

Hence $A_1 \times A_2$ is $\delta$-semi-open in $X_1 \times X_2$.

If $X = X_1 \times X_2$, $X_i$ being topological spaces and $A$ is $\delta$-semi-open in $X$, then it is not true, is general, that $A$ is a union of the form $A_1 \times A_2$ where $A_i$ is $\delta$-semi-open in $X_i$ for $i = 1, 2$.

**Example 2.26.** Let $X$ be the set of real numbers with usual topology and $A = \{(x, y) \mid 0 < x < 1, 0 < y < 1\} \cup \{(1, 1)\}$. Then $A$ is $\delta$-semi-open in $X \times X$ but it is not the product of any $\delta$-semi-open sets of $X$.

Let $\{X_\alpha \mid \alpha \in \Lambda\}$ be a family of spaces. We denote the product space $\prod\{X_\alpha \mid \alpha \in \Lambda\}$ by $\prod X_\alpha$. If $A_\alpha$ is a subset of $X_\alpha$ for each $\alpha \in \Lambda$, then $\prod A_\alpha$ denotes the set $\prod\{A_\alpha \mid \alpha \in \Lambda\}$ in $\prod X_\alpha$.

**Theorem 2.27.** *Let* $\{X_\alpha \mid \alpha \in \Lambda\}$ *be any family of spaces and* $A = \prod_{j=1}^n A_{\alpha_j} \times \prod_{\beta \neq \alpha_j} X_\beta$ *be a nonempty subset of the product space* $\prod X_\alpha$, *where* $n$ *is a positive integer. Then* $A$ *is* $\delta$*-semi-open in* $\prod X_\alpha$ *if and only if* $A_{\alpha_j}$ *is* $\delta$*-semi-open in* $X_{\alpha_j}$ *for each* $j = 1, 2, \ldots, n$.

## 3. Applications of δ-semi-open Sets

**Definition 3.1.** Let $A$ be a subset of a space $X$. A point $x \in X$ is said to be a *δ-semi-limit point* of $A$ if for each δ-semi-open set $U$ containing $x$, $U \cap (A \setminus \{x\}) \neq \varnothing$. The set of all δ-semi-limit points of $A$ is called a *δ-semi-derived set* of $A$ and is denoted by $\delta\text{-sd}(A)$.

It follows immediately from Definition 3.1 that every semi-limit point of $A$ is a δ-semi-limit point as well as limit point of $A$. But the converses are not true. Also the notions of a δ-semi-limit point and a limit point are independent of each other as are shown by Examples 3.2 and 3.3.

**Example 3.2.** Let $X$ be the real number space with its natural topology and $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$. Then $A \subset X$ and 0 is a limit-point of $A$. But 0 is neither a semi-limit point nor a δ-semi-limit point of $A$. Because $(-1, 0]$ is a δ-semi-open set containing 0, no point of $A$ belongs to $(-1, 0]$.

**Example 3.3.** Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$. Take $A = \{a, b, c\}$. Then $c$ is a δ-semi-limit point of $A$ but neither a semi-limit point nor a limit point of it.

**Theorem 3.4.** *For subsets $A$, $B$ of a space $X$, the following statements hold:*

(1) *If $A \subset B$, then $\delta\text{-sd}(A) \subset \delta\text{-sd}(B)$.*

(2) $\delta\text{-sd}(A) \cup \delta\text{-sd}(B) \subset \delta\text{-sd}(A \cup B)$ *and* $\delta\text{-sd}(A \cap B) \subset \delta\text{-sd}(A) \cap \delta\text{-sd}(B)$.

(3) $\delta\text{-sd}(\delta\text{-sd}(A)) \setminus A \subset \delta\text{-sd}(A)$.

(4) $\delta\text{-sd}(A \cup \delta\text{-sd}(A)) \subset A \cup \delta\text{-sd}(A)$.

**Proof.** (3) If $x \in \delta\text{-sd}(\delta\text{-sd}(A)) \setminus A$ and $U$ is a δ-semi-open set containing $x$, then $U \cap (\delta\text{-sd}(A) \setminus \{x\}) \neq \varnothing$. Let $y \in U \cap (\delta\text{-sd}(A) \setminus \{x\})$. Then since $y \in \delta\text{-sd}(A)$ and $y \in U$, $U \cap (A \setminus \{y\}) \neq \varnothing$. Let $z \in$

$U \cap (A \setminus \{y\})$. Then $z \neq x$ for $z \in A$ and $x \notin A$. Hence $U \cap (A \setminus \{x\}) \neq \varnothing$. Therefore $x \in \delta\text{-sd}(A)$.

(4) Let $x \in \delta\text{-sd}(A \cup \delta\text{-sd}(A))$. If $x \in A$, the result is obvious. So let $x \in \delta\text{-sd}(A \cup \delta\text{-sd}(A)) \setminus A$, then for δ-semi-open set $U$ containing $x$, $U \cap ((A \cup \delta\text{-sd}(A)) \setminus \{x\}) \neq \varnothing$. Thus $U \cap (A \setminus \{x\}) \neq \varnothing$ or $U \cap (\delta\text{-sd}(A) \setminus \{x\}) \neq \varnothing$. Now it follows similarly from (3) that $U \cap (A \setminus \{x\}) \neq \varnothing$. Hence $x \in \delta\text{-sd}(A)$. Therefore, in any case $\delta\text{-sd}(A \cup \delta\text{-sd}(A)) \subset A \cup \delta\text{-sd}(A)$.

Equalities may fail in (2) of the above theorem is shown by Examples 3.5 and 3.6.

**Example 3.5.** Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \varnothing, \{a\}, \{b, c\}, \{a, b, c\}\}$. Take $A = \{b, c\}$, $B = \{a, d\}$. Then $\delta\text{-sd}(A) = \{b, c\}$, $\delta\text{-sd}(B) = \varnothing$ and $\delta\text{-sd}(A \cup B) = \{b, c, d\}$. Hence $\delta\text{-sd}(A \cup B) \neq \delta\text{-sd}(A) \cup \delta\text{-sd}(B)$.

**Example 3.6.** Consider the space $(X, \tau)$ given in Example 3.3. Take $A = \{a, b, c\}$, $B = \{b, d, e\}$. Then $\delta\text{-sd}(A) \cap \delta\text{-sd}(B) \neq \delta\text{-sd}(A \cap B)$.

**Theorem 3.7.** $\delta\text{-sCl}(A) = A \cup \delta\text{-sd}(A)$.

**Proof.** Since $\delta\text{-sd}(A) \subset \delta\text{-sCl}(A)$ from Theorem 2.8(8) and Definition 3.1, $A \cup \delta\text{-sd}(A) \subset \delta\text{-sCl}(A)$. On the other hand, let $x \in \delta\text{-sCl}(A)$. If $x \in A$, then the proof is complete. If $x \notin A$, each δ-semi-open set $U$ containing $x$ intersects $A$ at a point distinct from $x$, so $x \in \delta\text{-sd}(A)$. Thus $\delta\text{-sCl}(A) \subset A \cup \delta\text{-sd}(A)$, which completes the proof.

**Corollary 3.8.** *A subset A is δ-semi-closed if and only if it contains the set of its δ-semi-limit points.*

**Definition 3.9.** A point $x \in X$ is said to be a *δ-semi-interior point* of $A$ if there exists a δ-semi-open set $U$ containing $x$ such that $U \subset A$. The set of all δ-semi-interior points of $A$ is said to be a *δ-semi-interior* of $A$.

Since δ-semi-open is semi-open, every δ-semi-interior point of $A$ is a semi-interior point of $A$. But the converses are not true, in general, as are shown by Example 3.10.

**Example 3.10.** Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$. Take $A = \{a, c, d\}$. Then $d$ is a semi-interior point of $A$ but not a $\delta$-semi-interior point of $A$.

**Theorem 3.11.** *For subsets A, B of a space X, the following statements are true:*

(1) $\delta\text{-}s\mathrm{Int}(A)$ *is the largest* $\delta$*-semi-open set contained in A.*

(2) *A is* $\delta$*-semi-open if and only if* $A = \delta\text{-}s\mathrm{Int}(A)$.

(3) $\delta\text{-}s\mathrm{Int}(\delta\text{-}s\mathrm{Int}(A)) = \delta\text{-}s\mathrm{Int}(A)$.

(4) $\delta\text{-}s\mathrm{Int}(A) = A \setminus \delta\text{-}s\mathrm{d}(X \setminus A)$.

(5) $X \setminus \delta\text{-}s\mathrm{Int}(A) = \delta\text{-}s\mathrm{Cl}(X \setminus A)$.

(6) $X \setminus \delta\text{-}s\mathrm{Cl}(A) = \delta\text{-}s\mathrm{Int}(X \setminus A)$.

(7) *If* $A \subset B$, *then* $\delta\text{-}s\mathrm{Int}(A) \subset \delta\text{-}s\mathrm{Int}(B)$.

(8) $\delta\text{-}s\mathrm{Int}(A) \cup \delta\text{-}s\mathrm{Int}(B) \subset \delta\text{-}s\mathrm{Int}(A \cup B)$.

(9) $\delta\text{-}s\mathrm{Int}(A) \cap \delta\text{-}s\mathrm{Int}(B) \supset \delta\text{-}s\mathrm{Int}(A \cap B)$.

**Proof.** (4) If $x \in A \setminus \delta\text{-}s\mathrm{d}(X \setminus A)$, then $x \notin \delta\text{-}s\mathrm{d}(X \setminus A)$ and so there exists a $\delta$-semi-open set $U$ containing $x$ such that $U \cap (X \setminus A) = \emptyset$. Then $x \in U \subset A$ and hence $x \in \delta\text{-}s\mathrm{Int}(A)$, i.e., $A \setminus \delta\text{-}s\mathrm{d}(X \setminus A) \subset \delta\text{-}s\mathrm{Int}(A)$. On the other hand, if $x \in \delta\text{-}s\mathrm{Int}(A)$, then $x \notin \delta\text{-}s\mathrm{d}(X \setminus A)$ for $\delta\text{-}s\mathrm{Int}(A)$ is $\delta$-semi-open and $\delta\text{-}s\mathrm{Int}(A) \cap (X \setminus A) = \emptyset$. Hence $\delta\text{-}s\mathrm{Int}(A) = A \setminus \delta\text{-}s\mathrm{d}(X \setminus A)$.

(5)
$$\begin{aligned} X \setminus \delta\text{-}s\mathrm{Int}(A) &= X \setminus (A \setminus \delta\text{-}s\mathrm{d}(X \setminus A)) \\ &= (X \setminus A) \cup \delta\text{-}s\mathrm{d}(X \setminus A) \\ &= \delta\text{-}s\mathrm{Cl}(X \setminus A). \end{aligned}$$

**Definition 3.12.** $\delta\text{-}s\mathrm{b}(A) = A \setminus \delta\text{-}s\mathrm{Int}(A)$ is said to be the $\delta$*-semi-border* of $A$.

**Theorem 3.13.** *For a subset $A$ of a space $X$, the following statements are true:*

(1) $\mathrm{sb}(A) \subset \delta\text{-}\mathrm{sb}(A)$ *where* $\mathrm{sb}(A)$ *denotes the semi-border of* $A$.

(2) $A = \delta\text{-}s\mathrm{Int}(A) \cup \delta\text{-}\mathrm{sb}(A)$.

(3) $\delta\text{-}s\mathrm{Int}(A) \cap \delta\text{-}\mathrm{sb}(A) = \varnothing$.

(4) *$A$ is $\delta$-semi-open set if and only if* $\delta\text{-}\mathrm{sb}(A) = \varnothing$.

(5) $\delta\text{-}\mathrm{sb}(\delta\text{-}s\mathrm{Int}(A)) = \varnothing$.

(6) $\delta\text{-}s\mathrm{Int}(\delta\text{-}\mathrm{sb}(A)) = \varnothing$.

(7) $\delta\text{-}\mathrm{sb}(\delta\text{-}\mathrm{sb}(A)) = \delta\text{-}\mathrm{sb}(A)$.

(8) $\delta\text{-}\mathrm{sb}(A) = A \cap \delta\text{-}s\mathrm{Cl}(X \setminus A)$.

(9) $\delta\text{-}\mathrm{sb}(A) = \delta\text{-}s\mathrm{d}(X \setminus A)$.

**Proof.** (6) If $x \in \delta\text{-}s\mathrm{Int}(\delta\text{-}\mathrm{sb}(A))$, then $x \in \delta\text{-}\mathrm{sb}(A)$. On the other hand, since $\delta\text{-}\mathrm{sb}(A) \subset A$, $x \in \delta\text{-}s\mathrm{Int}(\delta\text{-}\mathrm{sb}(A)) \subset \delta\text{-}s\mathrm{Int}(A)$. Hence $x \in \delta\text{-}s\mathrm{Int}(A) \cap \delta\text{-}\mathrm{sb}(A)$ which contradicts (3). Thus $\delta\text{-}s\mathrm{Int}(\delta\text{-}\mathrm{sb}(A)) = \varnothing$.

(8) $$\delta\text{-}\mathrm{sb}(A) = A \setminus \delta\text{-}s\mathrm{Int}(A) = A \setminus (X \setminus \delta\text{-}s\mathrm{Cl}(X \setminus A))$$
$$= A \cap \delta\text{-}s\mathrm{Cl}(X \setminus A).$$

(9) $$\delta\text{-}\mathrm{sb}(A) = A \setminus \delta\text{-}s\mathrm{Int}(A)$$
$$= A \setminus (A \setminus \delta\text{-}s\mathrm{d}(X \setminus A))$$
$$= \delta\text{-}s\mathrm{d}(X \setminus A).$$

**Definition 3.14.** $\delta\text{-}\mathrm{sf}(A) = \delta\text{-}s\mathrm{Cl}(A) \setminus \delta\text{-}s\mathrm{Int}(A)$ is said to be the *$\delta$-semi-frontier* of $A$.

**Theorem 3.15.** *For a subset $A$ of a space $X$, the following statements are true:*

(1) $\delta\text{-}\mathrm{sf}(A) \subset \mathrm{sf}(A)$ *where* $\mathrm{sf}(A)$ *denotes the semi-frontier of* $A$.

(2) $\delta\text{-}s\mathrm{Cl}(A) = \delta\text{-}s\mathrm{Int}(A) \cup \delta\text{-}s\mathrm{f}(A)$.

(3) $\delta\text{-}s\mathrm{Int}(A) \cap \delta\text{-}s\mathrm{f}(A) = \varnothing$.

(4) $\delta\text{-}s\mathrm{b}(A) \subset \delta\text{-}s\mathrm{f}(A)$.

(5) $\delta\text{-}s\mathrm{f}(A) = \delta\text{-}s\mathrm{b}(A) \cup \delta\text{-}s\mathrm{d}(A)$.

(6) *$A$ is δ-semi-open if and only if* $\delta\text{-}s\mathrm{f}(A) = \delta\text{-}s\mathrm{d}(A)$.

(7) $\delta\text{-}s\mathrm{f}(A) = \delta\text{-}s\mathrm{Cl}(A) \cap \delta\text{-}s\mathrm{Int}(X \setminus A)$.

(8) $\delta\text{-}s\mathrm{f}(A) = \delta\text{-}s\mathrm{f}(X \setminus A)$.

(9) $\delta\text{-}s\mathrm{f}(A)$ *is δ-semi-closed.*

(10) $\delta\text{-}s\mathrm{f}(\delta\text{-}s\mathrm{f}(A)) \subset \delta\text{-}s\mathrm{f}(A)$.

**Proof.** (5) Since $\delta\text{-}s\mathrm{Int}(A) \cup \delta\text{-}s\mathrm{f}(A) = \delta\text{-}s\mathrm{Int}(A) \cup \delta\text{-}s\mathrm{b}(A) \cup \delta\text{-}s\mathrm{d}(A)$, $\delta\text{-}s\mathrm{f}(A) = \delta\text{-}s\mathrm{b}(A) \cup \delta\text{-}s\mathrm{d}(A)$.

(7)
$$\begin{aligned}\delta\text{-}s\mathrm{f}(A) &= \delta\text{-}s\mathrm{Cl}(A) \setminus \delta\text{-}s\mathrm{Int}(A)\\ &= \delta\text{-}s\mathrm{Cl}(A) \cap \delta\text{-}s\mathrm{Cl}(X \setminus A)\\ &= \delta\text{-}s\mathrm{Cl}(A) \cap (X \setminus \delta\text{-}s\mathrm{Int}(A)).\end{aligned}$$

(9)
$$\begin{aligned}\delta\text{-}s\mathrm{Cl}(\delta\text{-}s\mathrm{f}(A)) &= \delta\text{-}s\mathrm{Cl}(\delta\text{-}s\mathrm{Cl}(A) \cap \delta\text{-}s\mathrm{Cl}(X \setminus A))\\ &\subset \delta\text{-}s\mathrm{Cl}(\delta\text{-}s\mathrm{Cl}(A)) \cap \delta\text{-}s\mathrm{Cl}(\delta\text{-}s\mathrm{Cl}(X \setminus A))\\ &= \delta\text{-}s\mathrm{Cl}(A) \cap \delta\text{-}s\mathrm{Cl}(X \setminus A) = \delta\text{-}s\mathrm{f}(A).\end{aligned}$$

Hence $\delta\text{-}s\mathrm{f}(A)$ is δ-semi-closed.

**Definition 3.16.** $\delta\text{-se}(A) = \delta\text{-}s\mathrm{Int}(X \setminus A)$ is said to be a *δ-semi-exterior* of $A$.

**Theorem 3.17.** *For subsets $A, B$ of a space $X$, the following statements are true:*

(1) $\delta\text{-se}(A) \subset \mathrm{se}(A)$ *where* $\mathrm{se}(A)$ *denotes the semi-exterior of* $A$.

(2) $\delta$-se$(A)$ *is* $\delta$-*semi-open*.

(3) $\delta\text{-se}(A) = \delta\text{-}s\mathrm{Int}(X \setminus A) = X \setminus \delta\text{-}s\mathrm{Cl}(A)$.

(4) $\delta\text{-se}(\delta\text{-se}(A)) = \delta\text{-}s\mathrm{Int}(\delta\text{-}s\mathrm{Cl}(A))$.

(5) *If* $A \subset B$, *then* $\delta\text{-se}(A) \supset \delta\text{-se}(B)$.

(6) $\delta\text{-se}(A \bigcup B) \subset \delta\text{-se}(A) \bigcap \delta\text{-se}(B)$.

(7) $\delta\text{-se}(A \bigcap B) \supset \delta\text{-se}(A) \bigcup \delta\text{-se}(B)$.

**Proof.** (4)
$$\begin{aligned}\delta\text{-se}(\delta\text{-se}(A)) &= \delta\text{-se}(X \setminus \delta\text{-}s\mathrm{Cl}(A)) \\ &= \delta\text{-}s\mathrm{Int}(X \setminus (X \setminus \delta\text{-}s\mathrm{Cl}(A))) \\ &= \delta\text{-}s\mathrm{Int}(\delta\text{-}s\mathrm{Cl}(A)).\end{aligned}$$

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# ON MINIMAL SUBGROUPS OF FINITE GROUPS

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## Abstract

This paper is devoted to discussing the structure of a finite group $G$ by using formation theory under the assumption that all minimal subgroups of $G$ are well-suited in $G$.

## 1. Introduction

All groups considered in this paper will be finite. We use conventional notions and notations, as in Huppert [5].

Recall that a minimal subgroup of a finite group is a subgroup of prime order. For the group of even order, it is also helpful to consider the cyclic subgroup of order 4. Of late there has been considerable interest in studying the structure of a finite group $G$, with the assumption that all minimal subgroups of $G$ are well-suited in the group. Ito has proved that if $G$ is a group of odd order and all minimal subgroups of $G$ lie in the center of $G$, then $G$ is nilpotent (see [5, p. 283]). An extension of Ito's result is the following statement. If for an odd prime $p$, every subgroup of order $p$ lies in the center of $G$, then $G$ is $p$-nilpotent. If all the elements of $G$ of order 2 or 4 lie in the center of $G$, then $G$ is 2-nilpotent (see [5, p. 435]). Buckley [4] proved that if $G$ is a group of odd order and all minimal subgroups of $G$ are normal in $G$, then $G$ is supersolvable. Later Shaalan

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[8] proved that if $G$ is a finite group and every cyclic subgroup of prime order or order 4 is $\pi$-quasinormal in $G$, then $G$ is supersolvable. Many scholars extend the results mentioned above using formation theory, such as in [1, 3, 11, 12].

We say, following Wang [9] that a subgroup $H$ of a group $G$ is $c$-normal in $G$ if there exists a normal subgroup $N$ of $G$ such that $G = HN$ and $H \cap N \le H_G = Cor_G(H)$. In [6], Li and Guo proved that if the finite group $G$ possesses a normal subgroup $N$ such that $G/N$ is supersolvable, and if every cyclic subgroup of prime order or order 4 of $N$ is $c$-normal in $G$, then $G$ is supersolvable. The main purpose of this paper is to extend this result using formation theory.

Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a *formation* provided that (i) if $G \in \mathcal{F}$ and $H \lhd G$, then $G/H \in \mathcal{F}$, and (ii) if $G/M$ and $G/N$ are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for normal subgroups $M$, $N$ of $G$. Each group has a smallest normal subgroup $N$ such that $G/N$ is in $\mathcal{F}$. This uniquely determined normal subgroup of $G$ is called the $\mathcal{F}$*- residual subgroup* of $G$ and will be denoted by $G^{\mathcal{F}}$. A formation $\mathcal{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$ (see [5, Ch. VI]). Throughout this paper $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a formation. Since a group $G$ is supersolvable iff $G/\Phi(G)$ is supersolvable (see [5, p. 713, Satz 8.6]), it follows that $\mathcal{U}$ is saturated.

**Definition.** Let $p$ be a prime and $G$ be a group. We define

$$\mathcal{P}_p(G) = \{x \mid x \in G, |x| = p\},$$

$$\mathcal{P}_4(G) = \{x \mid x \in G, |x| = 4\},$$

$$\mathcal{P}(G) = \cup_{p\in\pi(G)}\mathcal{P}_p(G),$$

$$\mathcal{P}^*(G) = \mathcal{P}_4(G) \cup \mathcal{P}(G).$$

Let $x$ be an element of $G$. We say that $x$ is *c-normal* in $G$ if $\langle x\rangle$ is $c$-normal in $G$.

**Lemma 1.** *Let $G$ be a group.*

(1) *If $H$ is normal in $G$, then $H$ is c-normal in $G$.*

(2) *If $H$ is c-normal in $G$ and $H \leq K \leq G$, then $H$ is c-normal in $K$.*

(3) *Let $K \lhd G$ and $K \leq H$. Then $H$ is c-normal in $G$ if and only if $H/K$ is c-normal in $G/K$.*

(4) *Let $\pi$ be a set of primes, $H$ be a normal $\pi'$-subgroup of $G$ and $T$ be a $\pi$-subgroup of $G$. If $T$ is c-normal in $G$, then $TH/H$ is c-normal in $G/H$.*

(5) *If $P$ is a minimal normal $p$-subgroup of $G$ and $x \in P$ is c-normal in $G$, then $P = \langle x \rangle$.*

**Proof.** (1), (2), (3) can be found in [9, Lemma 2.1], (4) is [6, Lemma 2.4], and (5) is [10, Lemma 2.2(1)].

**Lemma 2.** (see Theorem 1 and Proposition 1 of [3]).

*Let $\mathcal{F}$ be a saturated formation. Assume that $G$ is a group such that $G$ does not belong to $\mathcal{F}$ and there exists a maximal subgroup $M$ of $G$ such that $M \in \mathcal{F}$ and $G = MF(G)$. Then $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is an $\mathcal{F}$-eccentric chief factor of $G$. $G^{\mathcal{F}}$ is a $p$-group for some prime $p$, $G^{\mathcal{F}}$ has exponent $p$ if $p > 2$ and exponent at most 4 if $p = 2$. Moreover, $G^{\mathcal{F}}$ is either elementary abelian or $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ is an elementary abelian group.*

## 2. Results

**Theorem 1.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose $G$ is a group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$. If every generator of $\mathcal{P}^*(H)$ is c-normal in $G$, then $G \in \mathcal{F}$.*

**Proof.** Assume that the result is false and let $G$ be a counterexample of minimal order. Then $G$ does not belong to $\mathcal{F}$ and $1 \neq G^{\mathcal{F}} \leq H$. By [2, Theorem (3.5)], $G$ has a maximal subgroup $M$ such that $G/M_G$ does not

belong to $\mathcal{F}$ and $G = MF^*(G)$ where $F^*(G) = Soc(G \bmod \Phi(G))$. This implies $G = MG^{\mathcal{F}} = MH$ and $M/(M \cap H) \in \mathcal{F}$. Moreover every generator of $\mathcal{P}^*(M \cap N)$ is $c$-normal in $G$, so in $M$ by Lemma 1(2). Hence $M$ satisfies the hypothesis of the theorem. By the minimality of $G$, it follows that $M \in \mathcal{F}$. On the other hand, by [6, Theorem 3.4], $H$ is supersoluble. Consequently, $G^{\mathcal{F}}$ is soluble and then $G = MF(G)$. Applying Lemma 2, we have that $G^{\mathcal{F}}$ is a $p$-group for some prime $p$, $G^{\mathcal{F}}$ has exponent $p$ if $p > 2$ and exponent at most 4 if $p = 2$. $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a minimal normal $p$-subgroup of $G/(G^{\mathcal{F}})'$. Denote $\overline{G} = G/\Phi(G^{\mathcal{F}})$, we prove first following:

(*) For any $x \in G^{\mathcal{F}}$, $\overline{x} = x\Phi(G^{\mathcal{F}})$ is $c$-normal in $\overline{G}$.

Let $x \in G^{\mathcal{F}}$. Then $x$ has prime order $p$ or order 4. Our hypothesis yields that $\langle x \rangle$ is $c$-normal in $G$. By the definition, there exists a normal subgroup $K$ of $G$ such that $\langle x \rangle K = G$ with $\langle x \rangle \cap K \le \langle x \rangle_G$. Let $K_1 = G^{\mathcal{F}} \cap K$. Then $K_1 \triangleleft G$. Since $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a minimal normal $p$-subgroup of $G^{\mathcal{F}}/(G^{\mathcal{F}})'$, $K_1 \le (G^{\mathcal{F}})'$ or $K_1 = G^{\mathcal{F}}$. Assume that $K_1 \le (G^{\mathcal{F}})'$, then $G^{\mathcal{F}} = G^{\mathcal{F}} \cap G = G^{\mathcal{F}} \cap (\langle x \rangle K) = \langle x \rangle (G^{\mathcal{F}} \cap K) = \langle x \rangle K_1 = \langle x \rangle (G^{\mathcal{F}})' = \langle x \rangle \Phi(G^{\mathcal{F}}) = \langle x \rangle$. Thus $(G^{\mathcal{F}})' = 1$. Obviously (*) holds. Hence we can assume that $K_1 = G^{\mathcal{F}}$, then $G^{\mathcal{F}} \le K$. It follows that $\langle x \rangle = \langle x \rangle \cap K \triangleleft G$. Hence $\overline{G} = \langle \overline{x} \rangle \overline{K}$ and $\langle \overline{x} \rangle = \langle \overline{x} \rangle \cap \overline{K} \triangleleft \overline{G}$. By the definition of $c$-normal subgroup, (*) holds.

By Lemma 1(5), we have that $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a cyclic group. Since $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is $G$-isomorphic to $Soc(G/M_G)$ it follows that $G/M_G$ is supersoluble, a contradiction.

**Remark (a).** The above theorem does not hold for arbitrary formations. Let $\mathcal{F}$ be the formation composed of all groups $G$ such that

$G^{\mathcal{U}}$, the supersoluble residual, is elementary abelian. It is clear that $\mathcal{U} \subseteq \mathcal{F}$ but $\mathcal{F}$ is not saturated. Take $G = SL(2, 3)$ and $H = Z(G)$. Then $G/H \in \mathcal{F}$ and every generator of $\mathcal{P}^*(H) = H$ is $c$-normal in $G$. However, $G$ does not belong to $\mathcal{F}$.

(b) Theorem 1 is not true for saturated formations which do not contain the class of supersoluble groups. For example, if $\mathcal{F} = \mathcal{N}$ the saturated formation of all nilpotent groups, then the symmetric group of degree 3 has a normal subgroup $H$ of order 3 and $|G/H| = 2$. However, $G$ is not nilpotent.

**Theorem 2.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose $G$ is a group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$. If every element of $\mathcal{P}(H)$ is c-normal in $G$ and $G$ is 2-nilpotent, then $G \in \mathcal{F}$.*

**Proof.** Assume the result is false and let $G$ be a counterexample of minimal order. By Theorem 1, we have that 2 divides $|H|$. Since $G$ is 2-nilpotent, it follows that $H = PK$, where $P$ is a Sylow 2-subgroup of $H$ and $K$ is a normal Hall $2'$-subgroup of $H$. Then $K$ is characteristic in $H$. So $K$ is a normal subgroup of $G$. Suppose that $K \neq 1$ and consider the group $G/K$. $H/K$ is a normal subgroup of $G/K$ and $(G/K)/(H/K) \cong G/H \in \mathcal{F}$. Moreover, every minimal subgroup of $H/K$ is $c$-normal in $G/K$ by Lemma 1(4) and $G/K$ is 2-nilpotent. By the minimality of $G$, it follows that $G/K \in \mathcal{F}$. Since $K$ is of odd order, $G \in \mathcal{F}$ by Theorem 1, a contradiction. Therefore $K = 1$ and $H = P$ is a 2-group. Let $Q$ be the normal Hall $2'$-subgroup of $G$. Then $G/Q \in \mathcal{U} \subseteq \mathcal{F}$. Hence $G \cong G/(H \cap Q) \in \mathcal{F}$, a contradiction.

**Theorem 3.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose $G$ is a group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$. If every element of $\mathcal{P}(H)$ is c-normal in $G$ and $H$ has abelian Sylow 2-subgroup, then $G \in \mathcal{F}$.*

**Proof.** Assume the result is false and let $G$ be a counterexample of

minimal order. Then $1 \neq G^{\mathcal{F}} \leq H$. Since by [6, Theorem 3.4] $H$ is supersoluble, it follows that $G^{\mathcal{F}}$ is soluble. By [2, Theorem (3.5)], $G$ has a maximal subgroup $M$ such that $G/M_G$ does not belong to $\mathcal{F}$ and $G = MF^*(G)$ where $F^*(G) = Soc(G \bmod \Phi(G))$, the generalized Fitting subgroup of $G$. This implies $G = MG^{\mathcal{F}} = MH$ and $M/(M \cap H) \in \mathcal{F}$. Moreover, every element of $\mathcal{P}(M \cap H)$ is $c$-normal in $G$, so in $M$ by Lemma 1(2). Hence $M$ satisfies the hypothesis of the theorem. By the minimality of $G$, it follows that $M \in \mathcal{F}$. Moreover, $G = MF(G)$. So Lemma 2 applies. Then $G^{\mathcal{F}}$ is a $p$-group for some prime $p$. Consider two cases.

**Case 1.** 2 divides $|G^{\mathcal{F}}|$. In this case, $G^{\mathcal{F}}$ is a 2-group and $G^{\mathcal{F}}$ is abelian because $H$ has abelian Sylow 2-subgroups. In particular, $\Phi(G^{\mathcal{F}}) = 1$ by Lemma 2. This means that $G^{\mathcal{F}}$ is an elementary abelian 2-group. Then every element of $\mathcal{P}^*(G^{\mathcal{F}})$, i.e., $\mathcal{P}(G^{\mathcal{F}})$, is $c$-normal in $G$. Hence $G \in \mathcal{F}$ by Theorem 1, a contradiction.

**Case 2.** $G^{\mathcal{F}}$ is of odd order. In this case, $\mathcal{P}^*(G^{\mathcal{F}}) = \mathcal{P}(G^{\mathcal{F}})$. Thus $G \in \mathcal{F}$ by Theorem 1, a contradiction.

Recently, Li [7] has proved that if the finite group $G$ possesses a normal subgroup of odd order such that $G/N$ is supersolvable and if, for each Sylow subgroup $P$ of $N$, every minimal subgroup of $P$ is normal in $N_G(P)$, then $G$ is supersolvable. The following theorem is a generalization of this result in some extension.

**Theorem 4.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $H$ having a Sylow tower of supersoluble type such that $G/H \in \mathcal{F}$. If for each Sylow subgroup $P$ of $H$, every generator of $\mathcal{P}^*(P)$ is $c$-normal in $N_G(P)$, then $G$ belongs to $\mathcal{F}$.*

**Proof.** Assume the result is false and let $G$ be a counterexample of

minimal order. Among the normal subgroups $X$ of $G$ satisfying the hypothesis of the theorem, we choose $H$ with $|H|$ minimal.

Let $q$ be the largest prime dividing $|H|$. And let $Q$ be a Sylow $q$-subgroup of $H$. It is clear that $Q$ is a normal subgroup of $G$. Consider the group $G/Q$. Then $H/Q$ is a normal subgroup of $G/Q$ having a Sylow tower of supersoluble type and $(G/Q)/(H/Q) \in \mathcal{F}$. Now, if $R/Q$ is a Sylow $p$-subgroup of $H/Q$, then $p \neq q$ and there exists a Sylow $p$-subgroup $P$ of $H$ such that $R = PH$. Let $xQ$, $x \in P$, be a generator of $\mathcal{P}^*(R/Q)$. Then $x \in \mathcal{P}^*(P)$ is a generator of $\mathcal{P}^*(P)$. By the hypothesis, $x$ is $c$-normal in $N_G(P)$. So $xQ$ is $c$-normal in $N_G(P)Q/Q = N_G(R/Q)$ by Lemma 1(4). Consequently, $G/Q$ satisfies the hypothesis of the theorem. By the minimal choice of $G$, it follows that $G/Q \in \mathcal{F}$ and by the minimality of $H$, we have $H = Q$. Consequently, every generator of $\mathcal{P}^*(H)$ is $c$-normal in $N_G(Q) = G$. By Theorem 1, it follows that $G \in \mathcal{F}$, a contradiction.

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# STABILITY OF QUASIFRAMES WITH QUASIFRAME OPERATORS

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**Abstract**

This paper is concerned with some stability results for quasiframes with quasiframe operators. It is shown that from these, how the results can be applied to obtain a decomposition aspect. Also, some internal characterizations of quasiframe operators are derived.

## 1. Introduction

The use of non-orthogonal wavelets including frames in function spaces other than $L^2$ poses tough questions which we do not yet know how to resolve. Recently, Holub [10] developed a natural theoretical question on how far frames are away from Riesz bases using the notion of a preframe operator and a model theory. In particular, Aldroubi [1] has shown how one can construct any frame $(x_i)$ starting with one frame $(\phi_i)$, using established frame operator in terms of the space $\ell^2$.

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The aim of this paper is to define the quasiframe for a separable complex Hilbert space $H$ and quasiframe operator on $H$ and we pose questions motivated by known results in the case of frame based on the overcompleteness of the quasiframe in terms of quasiframe operator.

## 2. Quasiframe Operators with Quasiframes

Let $H$ denote a separable complex Hilbert space. A sequence of vectors $(x_n)_{n\in\mathrm{N}}$ is a *quasiframe* for $H$ if there exist $A, B > 0$ such that

$$A\| x \|^2 \le \sum_{n\in\mathrm{N}} | \rho_n \langle x, x_n \rangle |^2 \le B\| x \|^2, \quad x \in H, \rho = (\rho_n) \in \ell^2. \tag{1}$$

$A$ and $B$ are the quasiframe bounds, and a quasiframe is *tight* if $A = B$. A quasiframe is *exact* if it is no longer a quasiframe whenever any one of its elements is removed. A *quasiframe operator* of the quasiframe $(x_n)$ is a bounded linear operator $S : H \to H$ defined by

$$Sx = \sum_{n\in\mathrm{N}} k_n \langle x, x_n \rangle x_n, \quad k_n = | \rho_n |^2. \tag{2}$$

Let $(x_n)$ be a quasiframe for $H$. Then $L : H \to \ell^2$ defined by $Lx = (| \rho_n |^2 \langle x, x_n \rangle)$ is said to be *quasi-Bessel* if $L$ is a well-defined linear operator.

It should be noted by a straightforward computation that if $(x_n)$ is a quasiframe with quasi-Bessel operator $L$, then the elementary properties are derived: for $x \in H$, $\exists B > 0$, such that $\sum | \rho_n \langle x, x_n \rangle |^2 \le B\| x \|^2$ and for $c = (c_n) \in \ell^2$, $\exists x \equiv L^* c = \sum_{n\in\mathrm{N}} c_n x_n$ in $H$, where $L^* : \ell^2 \to H$ is the adjoint of $L$, e.g., see Proposition 3.3 in Benedetto-Walnut [2].

The following Lemma 2.1[2] implies that a quasiframe operator $S$ factors through $\ell^2$ by $L$ and $L^*$:

**Lemma 2.1.** *Let $(x_n)$ be a quasiframe for $H$ with quasi-Bessel operator $L$. Then $L^*L$ is continuous and $S = L^*L$.*

Let $S$ be a quasiframe operator on $H$ with $\ell^2$-quasiframe representation

$$Sx = \sum k_n \langle x, x_n \rangle x_n.$$

Then, since

$$\langle Sx, x \rangle = \sum k_n | \langle x, x_n \rangle |^2 \geq 0,$$

$S$ is a positive operator, written $S \geq 0$, and hence $S = S^*$.

Following Benedetto-Walnut [2], as with existing frames, we now reformulate some generalized stable decomposition results:

**Theorem 2.2.** *A sequence of vectors $(x_n)$ is a quasiframe for $H$ with quasiframe bounds $A$ and $B$ if and only if the quasiframe operator $S$ on $H$ is a topological isomorphism with norm bounds $\| S \| \leq B$ and $\| S^{-1} \| \leq A^{-1}$.*

**Proof.** Let the quasiframe $S : H \to H$ be a topological isomorphism with given norm conditions. Since $\| S \| \leq B$, we have

$$\langle Sx, x \rangle = \sum k_n | \langle x, x_n \rangle |^2 \leq B \| x \|^2 = \langle Bx, x \rangle,$$

$k_n = | \rho_n |^2$, by $\ell^2$-quasiframe representation of $S$. As a consequence the second inequality of (1) is proved. Next, the elementary functional analytic argument shows that $AI \leq S$ if and only if $S^{-1} \leq A^{-1} I$ ($I$ is the identity map) [2]. Thus, the first inequality $A \| x \|^2 \leq \sum k_n | \langle x, x_n \rangle |^2$ is easily derived.

Assume $(x_n)$ is a quasiframe for $H$ with quasiframe bounds $A$ and $B$. We now start by setting

$$S_N x = \sum_{n \leq N} k_n \langle x, x_n \rangle x_n.$$

Let $N \geq M$ and $x \in H$. By Hölder's inequality and the upper quasiframe bounds, we compute

$$\| S_N x - S_M x \|^2 = \left\langle \sum_{M<n\le N} k_n \langle x, x_n \rangle x_n, \sum_{M<n\le N} k_n \langle x, x_n \rangle x_n \right\rangle$$

$$\le \sup_{\| y \| \le 1} \left( \sum | \rho_n \langle x, x_n \rangle |^2 \right) \left( \sum | \rho_n \langle x_n, y \rangle |^2 \right)$$

$$\le \sup_{\| y \| \le 1} B \| y \|^2 \sum | \langle x, x_n \rangle |^2$$

$$= B \sum_{M<n\le N} | \langle x, x_n \rangle |^2 \to 0$$

as $M, N \to \infty$.

Thus, $\lim_{N\to\infty} S_N x \equiv Sx$ exists in $H$.

Similarly, computing as above, we have

$$\| Sx \|^2 = \sup_{\| y \| \le 1} \left| \sum k_n \langle x, x_n \rangle \langle x_n, y \rangle \right|^2$$

$$\le \sup_{\| y \| \le 1} B \| x \|^2 B \| y \|^2 = C \| x \|^2 \quad \text{for } C > 0,$$

which implies that $S$ is bounded.

For the injectivity, let $Sx = 0$. Then $\langle Sx, x \rangle = \sum k_n | \langle x, x_n \rangle |^2 = 0$. Thus, using the lower quasiframe bound yields $x = 0$.

Finally, we show the surjectivity of $S$ and continuity of $S^{-1}$ by using the Neumann expansion. Since the quasiframe hypothesis is equivalent to $AI \le S \le BI$, we have easily that

$$0 \le I - \frac{1}{B} S := P_1 \le I - \frac{A}{B} I = \frac{B-A}{B} I := P_2.$$

Since $P_1$ and $P_2$ are self-adjoint and $P_1 \le P_2$, we have $\| P_1 \| \le \| P_2 \|$. Thus, $\left\| I - \frac{1}{B} S \right\| \le 1$.

Therefore, the Neumann expansion asserts that the operator

$\left(\frac{1}{B}S\right)^{-1}$ on $H$ is bounded linear. As a consequence,

$$S^{-1} = \frac{1}{B}\left(I - \left(I - \frac{1}{B}S\right)\right)^{-1} = \frac{1}{B}\sum_{0}^{\infty}\left(I - \frac{1}{B}S\right)^{k}$$

is well-defined bounded linear on $H$, which completes the proof.

Also, immediate from the proof are:

**Corollary 2.1.** *Let $S$ be a quasiframe operator on $H$ with a quasiframe $(x_n)$ for $H$ with quasiframe bounds $A$ and $B$. Then $B^{-1}I \leq S^{-1} \leq A^{-1}I$.*

**Corollary 2.2.** *For all $x \in H$, as an overcomplete basis we have the following quasi-decompositions in $H$:*

$$x = \sum k_n \langle x,\, S^{-1}x_n \rangle\, x_n$$

*and*

$$x = \sum k_n \langle x,\, x_n \rangle\, S^{-1}x_n,$$

*where $(S^{-1}x_n)$ is the dual quasiframe of $(x_n)$ with quasiframe bounds $B^{-1}$ and $A^{-1}$.*

## 3. Application to Vector Measure

Most of the results on 2-summing operators including $\ell_w^2(H)$ and $\ell_s^2(H)$ collected in this section can be found in [8], [9] and [12]. Recall that a quasiframe operator $S$ on a separable complex Hilbert space $H$ is 2-summing if and only if $(Sx_n) \in \ell_s^2(H)$, whenever $(x_n) \in \ell_w^2(H)$. Any 2-summing quasiframe operator $S$ on $H$ is continuous. In fact, if this were false, then some stable quasiframe $(x_n)$ in $H$ would satisfy $\| Sx_n \| > 2^n$, $n \in \mathbb{N}$, and $(2^{-n}x_n)$ would be in $\ell_w^2(H)$ but $(2^{-n}Sx_n)$ would not be in $\ell_s^2(H)$ a contradiction. It is well known that if $S : H \to H$ is 2-summing,

then the linear operation $\hat{S} : \ell_w^2(H) \to \ell_s^2(H)$ defined by $\hat{S}(x_n) = (Sx_n)$ has a closed graph and is, therefore, bounded linear. The 2-summing norm $\pi_2(S)$ of $S$ is defined by $\pi_2(S) := \| \hat{S} \|$; the operator norm of $\hat{S}$.

The collection of $\hat{S}$ constructs a closed linear subspace of $BL(\ell_w^2(H), \ell_s^2(H))$ of all bounded linear operators acting on $\ell_w^2(H)$ to $\ell_s^2(H)$. It follows that the collection $QB(H)$ of all 2-summing quasiframe operators on $H$ constitutes a complete normed space in the norm $\pi_2$. We recall that if $S$ is in $QB(H)$, then

$$\left(\sum_{i=1}^{n} \| Sx_i \|^2\right)^{\frac{1}{2}} \le C \sup_{\| x \| \le 1} \left(\sum_{1}^{n} | \langle x, x_i \rangle |^2\right)^{\frac{1}{2}},$$

$$\pi_2(S) = \inf\left\{C > 0 : \left(\sum \| Sx_i \|^2\right)^{\frac{1}{2}} \le C \sup_{\| x \| \le 1} \left(\sum | \langle x, x_i \rangle |^2\right)^{\frac{1}{2}}\right\}.$$

The following reformulated result analogous to the Grothendiek-Pietsch Domination Theorem links measure theory to the theory of 2-summing quasiframe operators, where we recall probability measure on a compact subset $K$ of $H$ for every $\mu \in M(K)$ such that $\mu \ge 0$ and $\| \mu \| = \mu(K) = 1$.

**Theorem 3.1.** *Let $S$ be in $QB(H)$. Then there exists a regular Borel probability measure $\mu$ defined on the unit ball $B_H$ ($B_H = \{x \in H : \| x \| \le 1\}$ $= B_{H^*}$) such that*

$$\| Sx_n \| \le \pi_2(S) \left( \int_{B_H} | \langle x, x_n \rangle |^2 d\mu(x) \right)^{\frac{1}{2}}$$

*holds for each quasiframe $(x_n)$ in $H$.*

**Proof.** For finite $x_1, ..., x_n \in H$, we get $f_{x_1, ..., x_n} : B_H \to \mathbb{R}$ by

$$f_{x_1, ..., x_n}(x) = \pi_2(S) \sum_{1}^{n} | \langle x, x_k \rangle |^2 - \sum_{1}^{n} \| Sx_k \|^2.$$

Then each $f_{x_1, \ldots, x_n}$ is continuous on $B_H$ and the collection $P = \{f_{x_1, \ldots, x_n} \in C(B_H) \mid x_1, \ldots, x_n \in H\}$ is a convex cone in the linear span $C(B_H)$, each of whose members is somewhere nonnegative - this last fact being due to the 2-summing nature of $S$. Now $C$ is disjoint from the convex cone $N = \{f \in C(B_H) \mid f(x) < 0 \text{ for } x \in B_H\}$ and this latter cone has an interior. Thus, there is a nonzero continuous linear functional $\mu \in C(B_H)^*$, which is a regular Borel measure on $B_H$ such that

$$\int f d\mu = \mu(f) \leq 0 \leq \mu(g) = \int g d\mu, \qquad \text{for } f \in N,\ g \in C.$$

$\mu$ has the distinction of being nonpositive on strictly negative functions; therefore it is nonnegative on strictly positive functions, and it follows that $\mu$ is a nonnegative measure. Normalizing $\mu$ gives a probability measure. Also, $\mu$ is nonnegative on $C$; so $\int f_{x_n} d\mu \geq 0$ for $(x_n)$ in $H$. But this means that

$$\| Sx_n \|^2 \leq \pi_2^2(S) \int_{B_H} |\langle x, x_n \rangle|^2 d\mu(x).$$

It is clear that every existing frame operator is approximable and hence is the limit in operator norm of a sequence of finite rank operators and must be compact in the sense of A. Pietsch [12]. A similar argument is employed in the following, where Pietsch integral and its norm $\| \ \|_{pint}$ are taken from Diestel [8] and the finite sections $P_n\rho\,(\rho \in \ell^2)$ from Bu and Gupta [3].

**Theorem 3.2.** *Every quasiframe operator $S : H \to H$ is compact Pietsch integral. In this case, $\| S \| \leq \| S \|_{pint}$.*

**Proof.** Let $S$ be in $QB(H)$ with $\ell^2$-quasiframe representation

$$Sx = \sum_1^\infty k_n \langle x, x_n \rangle x_n$$

and let $S_n$ be the finite rank operator defined by

$$S_n x = \sum_{1}^{n} k_i \langle x,\, x_i \rangle\, x_i.$$

Now, we consider for $L \in H^{\beta}$, where $H^{\beta}$ is β-dual of $H$; $H^{\beta} = \{L \mid L; H \to \ell^2$ is quasi-Bessel, by $Lx = (|\rho_n|^2 \langle x,\, x_n \rangle)$, $\rho = (\rho_n) \in \ell^2\}$. Since

$$\| S - S_n \| = \sup_{\| x \| \le 1,\, \| L \| \le 1} \left| \sum_{i=n+1}^{\infty} |\rho_i|^2 \langle x,\, x_i \rangle \| Lx_i \|_{\ell^2} \right|$$

$$\le \sup_i \| x_i \| \| |\rho| - P_n |\rho| \| \sup_{\| L \| \le 1} \| (\| Lx_i \|) \|.$$

Result follows from the fact that $\| |\rho| - P_n |\rho| \| \to 0$, while the collection of $(\| Lx_i \|)_{\| L \| \le 1}$ is norm bounded in $\ell^2$. This proves the first assertion. Next, to see that $S$ is Pietsch integral, let $Sx = \sum_1^{\infty} k_n \langle x,\, x_n \rangle\, x_n$ with $\sum_1^{\infty} \| \rho \| \| x_i \| \| (\| Lx_i \|) \| \le 1$. As usual, define an $H$-valued vector measure $G$ on the Borel σ-algebra of $B_H$ in view of the self-duality of $H$.

$$G(M) = \sum_{1}^{\infty} \| \rho \| \| x_i \| \delta_{x_i}(M)\, x_i,$$

where $M$ is a Borel set, $x_i \in B_H$, and $\delta_{x_i}$ denotes the Dirac functional at $x_i$.

Obviously, $G$ is a σ-additive vector measure of bounded variation with

$$|G|(B_H) \le \sum_{1}^{\infty} \| \rho \| \| x_i \| \| (\| Lx_i \|) \|.$$

Moreover, for each $x \in H$, there exist $\rho \in \ell^2$ and $x_i \in B_H$ by Riesz Representation Theorem for which

$$Sx = \int_{B_H} \| \rho \| \langle x,\, x_i \rangle\, dG,$$

establishes the second.

Finally,

$$\| Sx \| = \left\| \int_{B_H} \| \rho \| \langle x, x_i \rangle \, dG \right\|$$

$$\leq \int_{B_H} \| \rho \| \| \langle x, x_i \rangle \| \, d|G|(B_H)$$

$$\leq \| \rho \| \| Lx \| \int_{B_H} d|G|$$

$$= \| \rho \| \| Lx \| |G|(B_H).$$

Hence, we have

$$\| S \| \leq \inf \| \rho \| \| L \| |G|(B_H) = \| S \|_{pint}.$$

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# ON A WEAK SEPARATION AXIOM

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## Abstract

In this paper, we introduce a new separation axiom by involving preopen sets called pre-$D_1$ which lies strictly between pre-$T_0$ and pre-$T_1$.

## 1. Introduction

Since the advent of the well-known separation axioms $T_0$, $T_1$, $T_2$ in topology, several of their generalizations have been introduced and investigated. For example, Maheshwari and Prasad [4] introduced and studied semi-$T_0$, semi-$T_1$ and semi-$T_2$ by changing open sets in $T_0$, $T_1$, $T_2$ to semi-open sets [3]. In 1989, Nour [8] introduced the notions of pre-$T_0$, pre-$T_1$ and pre-$T_2$ in the same manner by using preopen sets [6] instead of open sets. In 1997, Caldas [1] introduced and investigated a weaker separation axiom called semi-$D_1$ by using sD-sets. He showed that the separation axiom semi-$D_1$ is placed strictly between semi-$T_0$ and semi-$T_1$.

The object of this paper is to present the notion of pD-set and then

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introduce and characterize a new weak separation axiom called *pre-*$D_1$ by using pD-sets. It turns out that the separation axiom pre-$D_1$ lies strictly between pre-$T_0$ and pre-$T_1$.

## 2. Preliminaries

A subset $A$ of a topological space is called *preopen* [6] (resp. *semi-open* [3] and *α-open* [7]) if $A \subset Int(Cl(A))$ (resp. $A \subset Cl(Int(A))$ and $A \subset Int(Cl(Int(A)))$). The complement of a preopen set is called *preclosed*. The intersection of all preclosed sets containing a subset $A$ is called the *preclosure* [2] of $A$ and denoted by $pCl(A)$. The family of all preopen sets of $X$ will be denoted by $PO(X)$. For a point $x \in X$, we set $PO(X, x) = \{U \mid x \in U \in PO(X)\}$. The notions of pre-$T_0$, pre-$T_1$ and pre-$T_2$ are due to Nour [8] which are defined as follows:

**Definition 2.1.** A topological space $(X, \tau)$ is called *pre-*$T_0$ (resp. *semi-*$T_0$ [4]) if for any pair of distinct points $x$ and $y$ of $X$, there exists a preopen (resp. semi-open) set containing $x$ but not $y$ or a preopen (resp. semi-open) set containing $y$ but not $x$.

**Definition 2.2.** A topological space $(X, \tau)$ is called *pre-*$T_1$ (resp. *semi-*$T_1$ [4]) if for any pair of distinct points $x$ and $y$ of $X$, there exists a preopen (resp. semi-open) set containing $x$ but not $y$ and a preopen (resp. semi-open) set containing $y$ but not $x$.

**Definition 2.3.** A topological space $(X, \tau)$ is called *pre-*$T_2$ if for any pair of distinct points $x$ and $y$ of $X$, there exists disjoint preopen sets $G$ and $E$ containing $x$ and $y$, respectively.

Recall that a subset $G$ of a topological space $X$ is called a *semi-Difference set* [1] (briefly sD-sets) if there are two semi-open sets $U$ and $V$ in $X$ such that $U \neq X$ and $G = U \setminus V$. A topological space $(X, \tau)$ is called *semi-*$D_1$ [1] if for any pair of distinct points $x$ and $y$ of $X$, there exists a semi-open set containing $x$ but not $y$ and a semi-open set containing $y$ but not $x$. Caldas [1] showed that semi-$D_1$ is placed strictly between semi-$T_0$ and semi-$T_1$.

It should be noted that Nour [8] has shown that every $T_0$ (resp. $T_1$ and $T_2$) space is pre-$T_0$ (resp. pre-$T_1$ and pre-$T_2$) but the converses are not true: (1) Let $X = \{a, b, c\}$ and $\tau = \{X, \varnothing, \{b\}\}$, then $(X, \tau)$ is pre-$T_0$ but not $T_0$; (2) Let $X = \{a, b, c\}$ and $\tau = \{X, \varnothing, \{a, b\}\}$. Then $(X, \tau)$ is pre-$T_1$ but not $T_1$. The topological space $(X, \tau)$ is pre-$T_2$ but not $T_2$. It is obvious that if $(X, \tau)$ is pre-$T_1$ (resp. pre-$T_2$), then it is pre-$T_0$ (resp. pre-$T_1$) but the converses are not true.

**Definition 2.4.** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *preirresolute* [9] (resp. *precontinuous* [6]) if for every preopen (resp. open) set $V$ of $(Y, \sigma)$, the inverse image $f^{-1}(V)$ is preopen in $(X, \tau)$.

## 3. Pre-$D_1$ Spaces

We begin by introducing the following notions:

**Definition 3.1.** A subset $G$ of a topological space $X$ is called a *pre-Difference set* (briefly pD-set) if there are two $U, V \in PO(X)$ such that $U \neq X$ and $G = U \setminus V$.

It is obvious that every preopen set $U$ different from $X$ is a pD-set if $G = U$ and $V = \varnothing$ in the above definition. Take $X = \{a, b, c\}$ with topology $\tau = \{X, \varnothing, \{a\}\}$. Then $PO(X, \tau) = \{X, \varnothing, \{a\}, \{a, b\}, \{a, c\}\}$. For example, $\{b\}$ and $\{c\}$ are pD-sets but $\{b, c\}$ is not.

**Definition 3.2.** A topological space $(X, \tau)$ is called *pre-$D_0$* if for any distinct pair of points $x$ and $y$ of $X$ there exists a pD-set of $X$ containing $x$ but not $y$ or a pD-set of $X$ containing $y$ but not $x$.

**Definition 3.3.** A topological space $(X, \tau)$ is called *pre-$D_1$* if for any distinct pair of points $x$ and $y$ of $X$ there exists a pD-set of $X$ containing $x$ but not $y$ and a pD-set of $X$ containing $y$ but not $x$.

**Definition 3.4.** A topological space $(X, \tau)$ is called *pre-$D_2$* if for any distinct pair of points $x$ and $y$ of $X$ there exist disjoint pD-sets $G$ and $E$ of $X$ containing $x$ and $y$, respectively.

**Remark 3.1.** It follows from definitions that if the topological space $(X, \tau)$ is pre-$T_0$ (resp. pre-$T_1$ and pre-$T_2$), then it is pre-$D_0$ (resp. pre-$D_1$ and pre-$D_2$). It is also clear that if $(X, \tau)$ pre-$D_1$ (resp. pre-$D_2$), then it is pre-$D_0$ (resp. pre-$D_1$) but the converses are not true. Notice that the notions of pre-$D_1$ and semi-$D_1$ are independent of each other as the following examples show:

**Example 3.1.** Let $X = \{a, b, c\}$ and $\tau = \{X, \varnothing, \{a, b\}\}$. Then $(X, \tau)$ is pre-$D_1$ but not semi-$D_1$.

**Example 3.2.** Let $X = \{a, b, c\}$ and $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}\}$. Then $(X, \tau)$ is semi-$D_1$ but not pre-$D_1$.

**Theorem 3.1.** *For a topological space* $(X, \tau)$ *the following are satisfied*:

(1) $(X, \tau)$ *is pre-*$D_0$ *if and only if it is pre-*$T_0$.

(2) $(X, \tau)$ *is pre-*$D_1$ *if and only if it is pre-*$D_2$.

**Proof.** The sufficient conditions for (1) and (2) follow from Remark 3.1.

"Necessity condition" (1). Suppose that $(X, \tau)$ is pre-$D_0$ and for any distinct pair of points $x$ and $y$ of $X$ at least one belongs to a pD-set $G$. So we choose $x \in G$ and $y \notin G$. Assume that $G = U \setminus V$ for which $U \neq X$ and $U, V \in SO(X)$. It follows that $x \in U$. For the situation in which $y \notin G$ it raises two cases: (a) $y \notin U$, (b) $y \in U$ and $y \in V$. For the case (a), $X$ is pre-$T_0$ since $x \in U$ and $y \notin U$. The same is true for the case (b), since $y \in V$ but $x \notin V$.

"Necessity condition" (2). Assume that $X$ is pre-$D_1$. So by definition, for any distinct pair of points $x$ and $y$ of $X$ there exist pD-sets $G$ and $E$ such that $G$ containing $x$ but not $y$ and $E$ containing $y$ but not $x$. Suppose that $G = U \setminus V$ and $E = W \setminus N$ where $U, V, W, N \in SO(X)$. Since $E$ does not contain $x$, it follows that either $W$ does not contain $x$ or both $W$ and $N$ contain $x$. If $x \notin W$, then from $y \notin G$ either (i) $y \notin U$ or (ii)

$y \in U$ and $y \in V$. For the case (i), it follows from $x \in U \setminus V$ that $x \in U \setminus (V \cup W)$; and from $y \in W \setminus N$ follows that $y \in W \setminus (U \cup N)$. Now, we have $U \setminus (V \cup W)$ and $W \setminus (U \cup N)$ which are disjoint. For the case (ii), since $y \in U$ and $y \in V$, it follows that $x \in U \setminus V$ and $y \in V$. So $U \setminus V$ and $V$ are disjoint. Now if $x \in W$ and $x \in N$, it follows that $y \in W \setminus N$ and $x \in N$. Therefore, $W \setminus N$ and $N$ are disjoint. All this shows that $X$ is pre-$D_2$.

**Remark 3.2.** Notice that from Remark 3.1 and Theorem 3.1(1) it follows that every pre-$D_1$ space is pre-$T_0$ but the converse is not true. Take $X = \{a, b\}$ with topology $\tau = \{X, \{a\}, \varnothing\}$ as in [1, Example 2.7]. We have $PO(X, \tau) = \{X, \{a\}, \varnothing\}$. The space $(X, \tau)$ is $T_0$ and, therefore, pre-$T_0$ but not pre-$D_1$. Now take $X = \{a, b, c, d\}$ with topology $\sigma = \{X, \{a\}, \varnothing, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}$ as in [1, Example 2.8]. The space $(X, \sigma)$ is pre-$D_1$ but not pre-$T_1$.

We say that a point $x$ of a topological space $X$ which has $X$ as the only pre-neighborhood is called *common to all preclosed sets point* (briefly pcc).

**Theorem 3.2.** *For a pre-$T_0$ topological space $(X, \tau)$ the following are equivalent*:

(1) $(X, \tau)$ *is pre-$D_1$*;

(2) $(X, \tau)$ *has no pcc point*.

**Proof.** (1) $\Rightarrow$ (2) Since $(X, \tau)$ is pre-$D_1$, then each point $x$ of $X$ is contained in a pD-set $G = U \setminus V$ and, therefore, in $U$. By definition $U$ is different from $X$. This means that $x$ is not a pcc point.

(2) $\Rightarrow$ (1) If $X$ is a pre-$T_0$ space, then for each distinct pair of points $x$ and $y$ contained in $X$, at least one of them, $x$ (say) has a pre-neighborhood $U$ containing $x$ but not $y$. Thus $U$ which is different from $X$ is a pD-set. If $X$ has no pcc point, then $y$ is not a pcc point. So there exists a pre-neighborhood $V$ of $y$ which is different from $X$. Hence $V \setminus U$ contains $y$ but not $x$ and $V \setminus U$ is a pD-set. This means that $X$ is pre-$D_1$.

It is obvious that a pre-$T_0$ topological space $(X, \tau)$ is not pre-$D_1$ if and only if there is a unique pcc point in $X$. Why unique, because if $x$ and $y$ are both pcc points in $X$, then at least one of them, say $x$ has a pre-neighborhood $U$ containing $x$ but not $y$. Then it occurs a contradiction since $U$ is different from $X$.

## 4. Pre-symmetric Spaces

**Definition 4.1.** A topological space $(X, \tau)$ is called *pre-symmetric* if for $x$ and $y$ in $X$, $x \in pCl(\{y\})$ implies $y \in pCl(\{x\})$.

Recall that a subset $A$ of a topological space $(X, \tau)$ is called *pre generalized closed* [5] (briefly, $pg$-closed) if $pCl(A) \subseteq U$, whenever $A \subseteq U$ and $U \in PO(X, \tau)$.

**Theorem 4.1.** *A topological space $(X, \tau)$ is pre-symmetric if and only if $\{x\}$ is pg-closed for each $x \in X$.*

**Proof.** Let $x \in pCl(\{y\})$ but $y \notin pCl(\{x\})$. It follows that complement of $pCl(\{x\})$ contains $y$. Hence the set $\{y\}$ is a subset of the complement of $pCl(\{x\})$. This means that $pCl(\{y\})$ is a subset of the complement of $pCl(\{x\})$. Therefore, the complement of $pCl(\{x\})$ contains $x$ which is a contradiction.

Conversely, let $\{x\} \subset G \in PO(X, \tau)$ but $pCl(\{x\})$ is not a subset of $G$. This implies that $pCl(\{x\})$ and the complement of $G$ are not disjoint. Suppose $y$ belongs to their intersection. It follows that $x \in pCl(\{y\})$ which is a subset of the complement of $G$, and $x \notin G$ which is against what we have assumed.

**Corollary 4.1.** *If a topological space $(X, \tau)$ is a pre-$T_1$ space, then it is pre-symmetric.*

**Proof.** According to Nour [8, Theorem 1.19], singletons in a pre-$T_1$ space are preclosed and, therefore, $pg$-closed. Now, the result follows from Theorem 4.1.

**Corollary 4.2.** *For a topological space* $(X, \tau)$ *the following are equivalent:*

(1) $(X, \tau)$ *is pre-symmetric and pre-*$T_0$;

(2) $(X, \tau)$ *is pre-*$T_1$.

**Proof.** The sufficient condition is obvious.

Suppose that $x$ and $y$ are distinct points of $X$ and since $(X, \tau)$ is pre-$T_0$, let for some $G \in PO(X, \tau)$, $G$ be subset of the complement of $\{y\}$ and $x \in G$. It follows that $pCl(\{y\})$ does not contain $x$ and, therefore, $y$ does not belong to $pCl(\{x\})$. So there exist a preopen set $E$ of $X$ such that $E$ is a subset of the complement of $\{x\}$ and $(X, \tau)$ is pre-$T_1$.

**Theorem 4.2.** *For a pre-symmetric topological space* $(X, \tau)$ *the following are equivalent:*

(1) $(X, \tau)$ *is pre-*$T_0$;

(2) $(X, \tau)$ *is pre-*$D_1$;

(3) $(X, \tau)$ *is pre-*$T_{1/2}$;

(4) $(X, \tau)$ *is pre-*$T_1$.

**Proof.** Straightforward.

## 5. pD-sets and Preirresolute Functions

**Theorem 5.1.** *If* $f : X \to Y$ *is a preirresolute surjective function and* $G$ *is a pD-set in* $Y$*, then the inverse image of* $G$ *is a pD-set in* $X$.

**Proof.** Suppose that $G$ is a pD-set in $Y$. It follows that there exists $G, E \in PO(Y, \tau)$ such that $G = V \setminus U$ and $V$ is different from $Y$. Since $f$ is preirresolute, the inverse images of $U$ and $V$ are preopen in $X$. From the fact that $V$ is different from $Y$, it follows that the inverse image of $V$ is different from $X$. Therefore, $f^{-1}(G) = f^{-1}(U) \setminus f^{-1}(V)$ is a pD-set.

**Lemma 5.1** [8, Theorem 1.27]. *If* $f : X \to Y$ *is precontinuous and* $\alpha$*-open, then* $f^{-1}(V) \in PO(X)$ *for each* $V \in PO(Y)$.

**Theorem 5.2.** *Let $f : X \to Y$ be a precontinuous α-open surjection. If G is a pD-set in Y, then the inverse image of G is a pD-set in X.*

**Proof.** It follows from Lemma 5.1 and Theorem 5.1.

**Theorem 5.3.** *If $f : (X, \tau) \to (Y, \sigma)$ is preirresolute, bijective and $(Y, \sigma)$ is pre-$D_1$, then $(X, \tau)$ is pre-$D_1$.*

**Proof.** Let $Y$ be a pre-$D_1$ space and $x$ and $y$ be any pair of distinct points of $X$. By hypothesis, there exist pD-sets $G$ and $E$ of $Y$ containing $f(x)$ and $f(y)$, respectively such that $G$ does not contain $f(y)$ and $E$ does not contain $f(x)$. By Theorem 4.3, the inverse images of $G$ and $E$ are pD-sets in $X$ containing $x$ and $y$, respectively. This shows that $X$ is a pre-$D_1$ space.

**Theorem 5.4.** *A space X is pre-$D_1$ if and only if for each pair of distinct points x and y in X, there exists a preirresolute surjective function $f : X \to Y$, where Y is a pre-$D_1$ space such that $f(x)$ and $f(y)$ are distinct.*

**Proof.** We prove the sufficient condition since the necessity condition can be easily proved by taking, without loss of generality, the identity function defined on $X$ for each pair of distinct points of $X$ by which we obtain the result.

Suppose that $x \neq y$ and $x, y \in X$. Then, there exists preirresolute surjective function $f : X \to Y$, where $Y$ is a pre-$D_1$ space such that $f(x)$ and $f(y)$ are distinct. Thus, there exist disjoint pD-sets $G$ and $E$ in $Y$ such that $f(x) \in G$ and $f(y) \in E$. It follows from Theorem 4.3 that the inverse images of $G$ and $E$ are disjoint pD-sets of $X$ containing $x$ and $y$, respectively. Now the result follows from Theorem 3.1(2).

**Corollary 5.1.** *Let $\{x_i \mid i \in I\}$ be any family of topological spaces. If $x_i$ is pre-$D_1$ for each $i \in I$, then the product space $\prod X_i$ is pre-$D_1$.*

**Proof.** Suppose that $x_i$ and $y_i$ are any pair of distinct points of $\prod X_i$. Then there exists a $j \in I$ such that $x_j \neq y_j$. The projection

$P_j : \prod X_i \to X_j$ is an open continuous mapping and, therefore, preirresolute such that $P_j(x_i) \neq P_j(y_i)$. The product space $\prod X_i$ is pre-$D_1$ since $X_j$ is pre-$D_1$.

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# THE DETERMINANT OF SQUARE INTUITIONISTIC FUZZY MATRICES

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( Received May 25, 2001 )

Submitted by K. K. Azad

## Abstract

Using the idea of "intuitionistic fuzzy set" [Intuitionistic Fuzzy Sets, in: V. Ssurev, Ed., VII ITKR's Session, Sofia (June 1983 Central Sci. and Techn. Library, Bulg. Academy of Science), 1984; Fuzzy Sets and Systems 20 (1986), 87-96; Review and New Results on Intuitionistic Fuzzy Sets, Preprint IM-MFAIS-1-88, Sofia, 1988], we define the concept of intuitionistic fuzzy matrices as a natural generalization of fuzzy matrices. Also, we introduce and study the determinant theory for square intuitionistic fuzzy matrices.

## 1. Introduction

In 1965, Zadeh [7] introduced the concept of fuzzy sets. Since then various workers have contributed to the development of the fuzzy theory. Atanassov [1, 2, 3] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets.

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Using the idea of fuzzy sets, Kim et al. [4, 5, 6] studied fuzzy matrices as a generalization of matrices over the two element Boolean algebra. In this paper, using the idea of "intuitionistic fuzzy set", we define the notion of intuitionistic fuzzy matrices as a generalization of fuzzy matrices. Also, we introduce and investigate the determinant of square intuitionistic fuzzy matrices. Throughout this paper the notations used are standard.

## 2. Intuitionistic Fuzzy Matrices

An *intuitionistic fuzzy matrix* $\mathcal{A}$ is

$$\mathcal{A} = [(A, B)] = [(a_{ij}, b_{ij})],$$

where $A$ and $B$ are fuzzy matrices, and $a_{ij} + b_{ij} \leq 1$ for all $i, j$.

Obviously, every fuzzy matrix $A = [(a_{ij})]$ is an intuitionistic fuzzy matrix of the form $[(a_{ij}, 1 - a_{ij})]$.

Let $\mathcal{A} = [(a_{ij}, b_{ij})]$ and $\mathcal{B} = [(c_{ij}, d_{ij})]$ be $m \times n$ intuitionistic fuzzy matrices and let $\mathcal{C} = [(e_{ij}, f_{ij})]$ be an $n \times l$ intuitionistic fuzzy matrix. Then the matrix operations are defined by

(1) $\mathcal{A} + \mathcal{B} = [(a_{ij} \vee c_{ij}, b_{ij} \wedge d_{ij})]$.

(2) $\mathcal{AC} = \left[\left(\bigvee_{1 \leq k \leq n} (a_{ik} \wedge e_{kj}), \bigwedge_{1 \leq k \leq n} (b_{ik} \vee f_{kj})\right)\right]$.

(3) $\mathcal{A}^T = [(a_{ji}, b_{ji})]$ (the transpose of $\mathcal{A}$).

Let $J$ be an $n \times n$ fuzzy matrix that have all entries 1 and let $I$ be an $n \times n$ identity fuzzy matrix, and let $\mathcal{I} = [(I, J - I)]$. Then, by the simple calculation

$$\mathcal{AI} = \mathcal{IA} = \mathcal{A}.$$

Therefore, $\mathcal{I}$ is the *identity intuitionistic fuzzy matrix*. Let $P$ be an $n \times n$ permutation fuzzy matrix and $\mathcal{P} = [(P, J - P)]$. Then, by the simple calculation

$$\mathcal{PP}^T = \mathcal{P}^T\mathcal{P} = \mathcal{I}.$$

Therefore, $\mathcal{P}$ is a *permutation intuitionistic fuzzy matrix*.

**Theorem 2.1.** *Let $\mathcal{A} = [(A, B)]$ be an intuitionistic fuzzy matrix and let $\mathcal{P}$ be a permutation intuitionistic fuzzy matrix. Then $\mathcal{PA}$ is a row changed matrix of $\mathcal{A}$ and $\mathcal{AP}$ is a column changed matrix of $\mathcal{A}$.*

**Proof.** Suppose that $\mathcal{A}$ is an intuitionistic fuzzy matrix and $\mathcal{P}$ is a permutation intuitionistic fuzzy matrix which is generated by a permutation σ, where

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n-1 & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(i) & \cdots & \sigma(n-1) & \sigma(n) \end{pmatrix}.$$

Then

$$\mathcal{PA} = \left[\left(\bigvee_{1 \le k \le n} (p_{ik} \wedge a_{kj}), \bigwedge_{1 \le k \le n} (1 - p_{ik} \vee b_{kj})\right)\right]$$

$$= [(a_{\sigma(i)j}, b_{\sigma(i)j})].$$

Therefore, for any $i$, then $i$-th row of $\mathcal{PA}$ is a row of $\mathcal{A}$. The case of $\mathcal{AP}$ is similar to the above proof.

## 3. The Determinant of Square Intuitionistic Fuzzy Matrices

**Definition 3.1.** The determinant $|\mathcal{A}|$ of an $n \times n$ intuitionistic fuzzy matrix $\mathcal{A} = [(A, B)]$ is defined as follows:

$$|\mathcal{A}| = \left[\left(\bigvee_{\sigma \in S_n} a_{1\sigma(1)} \wedge \cdots \wedge a_{n\sigma(n)}, \bigwedge_{\sigma \in S_n} b_{1\sigma(1)} \vee \cdots \vee b_{n\sigma(n)}\right)\right],$$

where $S_n$ denotes the symmetric group of all permutations of the indices $(1, 2, \ldots, n)$.

**Example 3.2.** Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two fuzzy matrices such that

$$A = \begin{bmatrix} 0.5 & 0.3 \\ 0.6 & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 & 0.6 \\ 0.1 & 0.8 \end{bmatrix}.$$

Since $a_{ij} + b_{ij} \le 1$ for all $i, j$,

$$\mathcal{A} = [(A, B)] = \begin{bmatrix} (0.5, 0.4) & (0.3, 0.6) \\ (0.6, 0.1) & (0.2, 0.8) \end{bmatrix}$$

is a $2 \times 2$ intuitionistic fuzzy matrix. We calculate the determinant $|\mathcal{A}|$ as follows:

$$
\begin{aligned}
|\mathcal{A}| &= |[(A, B)]| \\
&= (\{0.5 \wedge 0.2\} \vee \{0.3 \wedge 0.6\}, \{0.4 \vee 0.8\} \wedge \{0.6 \vee 0.1\}) \\
&= (0.2 \vee 0.3,\ 0.8 \wedge 0.6) \\
&= (0.3,\ 0.6).
\end{aligned}
$$

**Theorem 3.3.** *If an intuitionistic fuzzy matrix $\mathcal{C}$ is obtained from an $n \times n$ intuitionistic fuzzy matrix $\mathcal{A} = [(A, B)]$ by multiplying $\mathcal{A}$ by $k \in (0, 1]$, then $|\mathcal{C}| = k|\mathcal{A}|$.*

**Proof.** Suppose that $\mathcal{C} = [(c_{ij}, d_{ij})] = [(ka_{ij}, kb_{ij})]$. Then

$$
\begin{aligned}
|\mathcal{C}| &= \left[\left(\bigvee_{\sigma \in S_n} c_{1\sigma(1)} \wedge \cdots \wedge c_{n\sigma(n)},\ \bigwedge_{\sigma \in S_n} d_{1\sigma(1)} \vee \cdots \vee d_{n\sigma(n)}\right)\right] \\
&= \left[\left(\bigvee_{\sigma \in S_n} ka_{1\sigma(1)} \wedge \cdots \wedge ka_{n\sigma(n)},\ \bigwedge_{\sigma \in S_n} kb_{1\sigma(1)} \vee \cdots \vee kb_{n\sigma(n)}\right)\right] \\
&= \left[\left(k \bigvee_{\sigma \in S_n} a_{1\sigma(1)} \wedge \cdots \wedge a_{n\sigma(n)},\ k \bigwedge_{\sigma \in S_n} b_{1\sigma(1)} \vee \cdots \vee b_{n\sigma(n)}\right)\right] \\
&= [(k|A|,\ k|B|)] \\
&= k[(|A|,\ |B|)] \\
&= k|\mathcal{A}|.
\end{aligned}
$$

**Theorem 3.4.** *Let $\mathcal{A} = [(A, B)]$ be an $n \times n$ intuitionistic fuzzy matrix. Then*

$$\det(\mathcal{I}_{ij}\mathcal{A}) = \det(\mathcal{A}) = \det(\mathcal{A}\mathcal{I}_{ij}),$$

*where $\mathcal{I}_{ij}$ is a permutation intuitionistic fuzzy matrix which is obtained*

*from the identity intuitionistic fuzzy matrix by interchanging row i and row j.*

**Proof.** Let $\mathcal{I}_{ij}\mathcal{A} = [(c_{ij}, d_{ij})]$. Then for any $i$, $j$, the $i$-th ($j$-th) row of $\mathcal{I}_{ij}\mathcal{A}$ is the $j$-th ($i$-th, respectively) row of $\mathcal{A}$. In fact, $\mathcal{I}_{ij}$ is a permutation intuitionistic fuzzy matrix which is generated by a permutation

$$\begin{pmatrix} i & j \\ j & i \end{pmatrix}.$$

Since, for any permutation $\sigma \in S_n$,

$$\begin{pmatrix} i & j \\ j & i \end{pmatrix}\sigma = \tau \in S_n,$$

$$|\mathcal{I}_{ij}\mathcal{A}| = \left[\left(\bigvee_{\sigma\in S_n} c_{1\sigma(1)} \wedge \cdots \wedge c_{n\sigma(n)}, \bigwedge_{\sigma\in S_n} d_{1\sigma(1)} \vee \cdots \vee d_{n\sigma(n)}\right)\right]$$

$$= \left[\left(\bigvee_{\tau\in S_n} a_{1\tau(1)} \wedge \cdots \wedge a_{n\tau(n)}, \bigwedge_{\tau\in S_n} d_{1\tau(1)} \vee \cdots \vee d_{n\tau(n)}\right)\right]$$

$$= |\mathcal{A}|.$$

The case of $\mathcal{A}\mathcal{I}_{ij}$ is similar to the above proof.

In fact, any permutation intuitionistic fuzzy matrix is the product of $\mathcal{I}_{ij}$'s for some $i$, $j$. Therefore, we have the following corollary:

**Corollary 3.5.** *Let* $\mathcal{A} = [(A, B)]$ *be an* $n \times n$ *intuitionistic fuzzy matrix. Then*

$$\det(\mathcal{P}\mathcal{A}\mathcal{Q}) = \det(\mathcal{A}),$$

*where* $\mathcal{P}$ *and* $\mathcal{Q}$ *are any permutation intuitionistic fuzzy matrices.*

From the Corollary 3.5, we know that $\det(\mathcal{P}\mathcal{A}) = \det(\mathcal{P})\det(\mathcal{A})$, where $\mathcal{P}$ is a permutation intuitionistic fuzzy matrix and $\mathcal{A}$ is any intuitionistic fuzzy matrix. In general, $\det(\mathcal{A}\mathcal{B}) \neq \det(\mathcal{A})\det(\mathcal{B})$ for any intuitionistic fuzzy matrices $\mathcal{A}$, $\mathcal{B}$.

**Example 3.6.** Let

$$\mathcal{A} = \begin{bmatrix} (0.14,\ 0.2) & (0.25,\ 0.18) \\ (0.12,\ 0.15) & (0.17,\ 0.1) \end{bmatrix} \text{ and } \mathcal{B} = \begin{bmatrix} (0.5,\ 0.5) & (0.3,\ 0.45) \\ (0.2,\ 0.4) & (0.16,\ 0.3) \end{bmatrix}$$

be intuitionistic fuzzy matrices. Then

$$\mathcal{A}\mathcal{B} = \begin{bmatrix} (0.2,\ 0.4) & (0.16,\ 0.3) \\ (0.17,\ 0.4) & (0.16,\ 0.3) \end{bmatrix}.$$

Thus $\det(\mathcal{A}) = (0.14, 0.18)$, $\det(\mathcal{B}) = (0.2, 0.45)$, $\det(\mathcal{A})\det(\mathcal{B}) = (0.14, 0.45)$ and $\det(\mathcal{A}\mathcal{B}) = (0.16, 0.4)$. Therefore, $\det(\mathcal{A})\det(\mathcal{B}) \neq \det(\mathcal{A}\mathcal{B})$.

**Theorem 3.7.** *Let $\mathcal{A} = [(A, B)]$ be an $n \times n$ intuitionistic fuzzy matrix. Then*

$$\det(\mathcal{A}) = \det(\mathcal{A}^T),$$

*where $\mathcal{A}^T$ denotes the transpose of $\mathcal{A}$.*

**Proof.** Let $\mathcal{A}^T = [(c_{ij}, d_{ij})]$. Then $\mathcal{A}^T = [(A^T, B^T)] = [(a_{ji}, b_{ji})]$. Since each permutation $\sigma$ is a one-to-one function, we have

$$|\mathcal{A}^T| = \left[\left(\bigvee_{\sigma \in S_n} c_{1\sigma(1)} \wedge \cdots \wedge c_{n\sigma(n)},\ \bigwedge_{\sigma \in S_n} d_{1\sigma(1)} \vee \cdots \vee d_{n\sigma(n)}\right)\right]$$

$$= \left[\left(\bigvee_{\sigma \in S_n} a_{\sigma(1)1} \wedge \cdots \wedge a_{\sigma(n)n},\ \bigwedge_{\sigma \in S_n} b_{\sigma(1)1} \vee \cdots \vee b_{\sigma(n)n}\right)\right]$$

$$= \left[\left(\bigvee_{\tau \in S_n} a_{1\tau(1)} \wedge \cdots \wedge a_{n\tau(n)},\ \bigwedge_{\tau \in S_n} b_{1\tau(1)} \vee \cdots \vee b_{n\tau(n)}\right)\right],$$

where the permutation $\tau$ is induced by the rearrangement of each $\sigma$ in $S_n$,

$$= |\mathcal{A}|.$$

**Theorem 3.8.** *Let $\mathcal{A} = [(A, B)]$ be an $n \times n$ intuitionistic fuzzy matrix. If $\mathcal{A}$ contains a zero row (column) in $A$ and a one row (column) in $B$, then $|\mathcal{A}| = (0, 1)$.*

**Proof.** Each term in $|A|$ of $|\mathcal{A} = [(A, B)]|$ contains a factor of each row (column) and hence a factor of zero row (column). Thus each term of $|A|$ is equal to zero, and consequently $|A| = 0$, and each term in $|B|$ of $|\mathcal{A} = [(A, B)]|$ contains a factor of each row (column) and hence contains a factor of one row (column). Thus each term of $|B|$ is equal to one, and consequently $|B| = 1$. Therefore, $|\mathcal{A}| = (0, 1)$.

The following theorem easily follows from Theorem 3.8 and is similar to a fundamental result in combinatorial matrix theory known as the Frobenius-Konig theorem.

A *diagonal* of an $n \times n$ matrix $A$ is a set of $n$ entries of $A$, no two of which lie on the same row or column.

**Theorem 3.9.** *Let* $\mathcal{A} = [(A, B)]$ *be an* $n \times n$ *intuitionistic fuzzy matrix. A necessary and sufficient condition for every diagonal of* $A(B)$ *to contain a zero (one, respectively) is that* $A(B)$ *contains an* $r \times (n - r + 1)$ *zero (one, respectively) submatrix.*

**Corollary 3.10.** *Let* $\mathcal{A} = [(A, B)]$ *be an* $n \times n$ *intuitionistic fuzzy matrix. Then, the determinant of* $\mathcal{A}$, $|\mathcal{A}|$, *is equal to* $(0, 1)$ *if and only if* $A(B)$ *contains an* $r \times (n - r + 1)$ *zero (one, respectively) submatrix.*

We define that if the entries $a_{ij}$ of $n \times n$ intuitionistic fuzzy matrix $\mathcal{A} = [(A, B)]$ are all zero, where $i > j$, and the entries $b_{ij}$ of $\mathcal{A} = [(A, B)]$ are all one where $i > j$, then $\mathcal{A}$ is an *upper triangular matrix*, and if the entries $a_{ij}$ of $\mathcal{A} = [(A, B)]$ are all zero where $i < j$ and the entries $b_{ij}$ of $\mathcal{A} = [(A, B)]$ are all one where $i < j$, then $\mathcal{A}$ is a *lower triangular matrix*. We call simply *triangular matrix* instead of upper or lower triangular matrix.

**Theorem 3.11.** *Let* $\mathcal{A} = [(A, B)]$ *be an* $n \times n$ *intuitionistic fuzzy matrix. If* $\mathcal{A}$ *is triangular, then the determinant of* $\mathcal{A}$,

$$|\mathcal{A}| = \left[\left(\bigwedge_{1 \le i \le n} a_{ii}, \bigvee_{1 \le i \le n} b_{ii}\right)\right].$$

**Proof.** Suppose that $\mathcal{A} = [(A, B)]$ is lower triangular. We consider the terms of $|\mathcal{A}|$ that

$$t_a = \bigwedge_{1 \le i \le n} a_{i\sigma(i)}, \quad t_b = \bigvee_{1 \le i \le n} b_{i\sigma(i)}.$$

Let $\sigma(1) \neq 1$. Then $1 < \sigma(1)$ and so $a_{1\sigma(1)} = 0$, $b_{1\sigma(1)} = 1$. This means that $t_a = 0$, $t_b = 1$ if $\sigma(1) \neq 1$. Now, let $\sigma(1) = 1$ and $\sigma(2) \neq 2$. Then $2 < \sigma(2)$ and $a_{2\sigma(2)} = 0$, $b_{2\sigma(2)} = 1$ and $t_a = 0$, $t_b = 1$. This means that $t_a = 0$, $t_b = 1$ if $\sigma(1) \neq 1$ or $\sigma(2) \neq 2$. Therefore, in this method, we know that each of the terms $t_a$, $t_b$ for $\sigma(1) \neq 1$ or $\sigma(2) \neq 2, \ldots, \sigma(n) \neq n$ must be zero, one respectively. Consequently,

$$|\mathcal{A}| = \left[\left(\bigwedge_{1 \le i \le n} a_{ii}, \bigvee_{1 \le i \le n} b_{ii}\right)\right].$$

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# REMARKS ON TWO QUESTIONS OF MATLIS

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## Abstract

It is shown that if $S$ is the socle of a regular $V$-ring $R$, then the selfinjectivity of the rings $R$ and $\frac{R}{S}$ are equivalent if and only if $S$ is finitely generated. It is observed that a ring $R$ is Artin semisimple if and only if every $R$-module is $\aleph_0$-injective.

## Introduction

E. Matlis in [8, Remark 1], has raised the following two questions:

1. Let $I$ be an infinite set. For each $i \in I$, let $F_i$ be a field and let $F = \prod_{i \in I} F_i$, $J = \sum_{i \in I} \oplus F_i$. Is $\frac{F}{J}$ a selfinjective ring?

2. Let $R$ be any commutative regular ring that is not a finite direct sum of fields, and let $S$ be the socle of $R$ (i.e., the sum of all the minimal ideals in $R$). Is the ring $\frac{R}{S}$ selfinjective?

The first question was answered by Karamzadeh in [6] by showing that $\frac{F}{J}$ is never selfinjective (in fact, two different proofs were given).

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This immediately gives a negative answer to the second question. Therefore, it is natural to ask if the rings in the above questions might enjoy a weaker form of selfinjectivity, or it seems reasonable to see when is the ring $\frac{R}{I}$ in the second question a selfinjective ring? Our main purpose of this article is to settle these questions. In [7], it is shown that the ring $\frac{F}{I}$ of the first question is $\aleph_0$-selfinjective. We extend this by observing that if $R$ is any regular right $\aleph_0$-selfinjective ring, then $\frac{R}{I}$ is always right $\aleph_0$-selfinjective, where $I$ is any ideal of $R$, a fact which is a lemma in [5]. We recall that by a well-known result of Kaplansky, see [1], a commutative regular ring is a $V$-ring (i.e., simple modules are injective), this suggests that we may consider the ring in the second question to be a regular $V$-ring. Motivated by a result in [2], namely, $C(X)$, the ring of real continuous functions modulo its socle is selfinjective if and only if $X$ is an extremally disconnected $P$-space with only a finite number of isolated points, we show that for a regular $V$-ring $R$ the selfinjectivity of rings $R$ and $\frac{R}{S}$ are equivalent if and only if the socle $S$ is finitely generated as a right ideal. We also observe that the well-known theorem of Osofsky which says $R$ is Artin semisimple if and only if every cyclic $R$-module is injective fails for $\aleph_0$-injectivity, but we show that $R$ becomes Artin semisimple if and only if every $R$-module is $\aleph_0$-injective.

All rings in this article are associative with identity and modules are unital right modules. An $R$-module $M$ is said to be *injective* ($\aleph_0$*-injective*) if every homomorphism from a right ideal (countably generated right ideal) of $R$ into $M$ can be extended to $R$. By a regular ring $R$, we mean von Neumann regular, i.e., if $a \in R$, then $\exists b \in R$ with $a = aba$.

**When is a regular ring modulo its socle, a selfinjective ring?**

We begin with the following proposition which is in [5, Lemma II.14.11], but we give a proof for the sake of completeness.

**Proposition 1.** *Let $R$ be a regular, right $\aleph_0$-selfinjective ring and suppose $I$ is a two-sided ideal of $R$, then $\frac{R}{I}$ is a right $\aleph_0$-selfinjective ring.*

**Proof.** Let $\frac{A}{I}$ be a countably generated right ideal in $\frac{R}{I}$ and $f : \frac{A}{I} \to \frac{R}{I}$ be an $\frac{R}{I}$-homomorphism. Let $\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n, \ldots,$ generate $\frac{A}{I}$, then since $R$ is regular, we have $\sum_1^\infty a_n R = \sum_1^\infty \oplus\, e_n R$, where $e_i$, $i = 1, 2, 3, \ldots, n, \ldots$ are orthogonal idempotents, see [3], i.e., $\sum \oplus\, \overline{e}_n \frac{R}{I} = \sum \overline{a}_n \frac{R}{I}$. Now for each $n$ we have $f(\overline{e}_n) = f(\overline{e}_n)\overline{e}_n$, hence $f(\overline{e}_n) = \overline{b}_n$ for some $b_n \in \mathrm{Re}_n$. But there is an $R$-homomorphism $g : \sum_1^\infty \oplus\, e_n R \to R$ such that $g(e_n) = b_n$, for all $n$. Now by the $\aleph_0$-selfinjectivity of $R$, there exists $r \in R$ with $g(e_n) = re_n = b_n$, $\forall n$, i.e., $f(\overline{e}_n) = \overline{b}_n = \overline{r}\,\overline{e}_n$, which shows that $f$ can be extended to $\frac{R}{I}$.

The next result is well-known if $\aleph_0$-injectivity is replaced by injectivity, see [1] or [11, Corollary 23.7]. The following proof works also for injectivity.

**Lemma 2.** *Let $R$ be a ring and $I$ be a two-sided ideal of $R$. Then whenever $\frac{R}{I}$ is $\aleph_0$-injective as an $R$-module, it is $\aleph_0$-selfinjective ring and the converse holds if $R$ is regular.*

**Proof.** Let $\frac{R}{I}$ be $\aleph_0$-injective as an $R$-module and $f : \frac{A}{I} \to \frac{R}{I}$ be an $\frac{R}{I}$-homomorphism, where $\frac{A}{I}$ is countably generated right ideal. Thus, we may put $A = B + I$, where $B = \sum_1^\infty a_i R$ and then, let $f(a_i + I) = a_i' + I$, where $a_i' \in R$. Now define $g : B \to \frac{R}{I}$ by $g(a_i) = f(a_i + I) =$

$a_i' + I$, then by our hypothesis $g$ can be extended to $R$, i.e., there exists $r + I \in \frac{R}{I}$ with $g(a_i) = (r + I)a_i = a_i' + I = ra_i + I$, which means that $f$ is extended to $\frac{R}{I}$. Conversely, let $\frac{R}{I}$ be right $\aleph_0$-selfinjective and let $f : A \to \frac{R}{I}$ be an $R$-homomorphism, where $A$ is a countably generated right ideal of $R$. First, we claim that $f(A \cap I) = 0$. To see this, it suffices to show that $f(e) = 0$, where $e^2 = e \in A \cap I$ (Note: $A \cap I$ is generated by idempotents) but clearly we have $f(e) = f(e)e \in \frac{R}{I}I = 0$. In as much as $\frac{A+I}{I} \cong \frac{A}{A \cap I}$ and $f(A \cap I) = 0$, we infer that $f$ induces an $R$-homomorphism $\bar{f} : \frac{A+I}{I} \to \frac{R}{I}$. Clearly, $\bar{f}$ can be extended to $\frac{R}{I}$. This means that there exists $r + I \in \frac{R}{I}$ with $\bar{f}(a + I) = (r + I)(a + I) = ra + I$, i.e., $f(a) = (r + I)a$ and we are through.

It is a well-known theorem of Osofsky, see [9] that if every cyclic $R$-module is injective then $R$ is Artin semisimple. The next result shows that in Osofsky's result injectivity cannot be replaced by $\aleph_0$-injectivity.

**Corollary 3.** *If every cyclic $R$-module is $\aleph_0$-injective, then $R$ is regular $\aleph_0$-selfinjective ring and the converse is true if $R$ is commutative.*

**Proof.** For the if part, it suffices to show that $R$ is regular. To this end, let $a \in R$ and define $f : aR \to \frac{R}{A}$, $f(ab) = b + A$ where $A = Ann(a) = \{r \in R : ar = 0\}$, then $f$ can be extended to $R$, i.e., there exists $c + A \in \frac{R}{A}$ with $f(a) = (c + A)a = 1 + A$. Thus $ca - 1 \in A$, i.e., $a(ca - 1) = 0$ which means $a = aca$. Conversely, by Proposition 1 and Lemma 2, we are through.

It is well-known that a ring $R$ is right Noetherian if and only if every

direct sum of injective modules is injective, see [4, Theorem 4.19]. The same proof yields the following:

**Theorem 4** (Papp-Bass). *A ring $R$ is right Noetherian if and only if every direct sum of $\aleph_0$-injective $R$-modules is $\aleph_0$-injective.*

We now have the following:

**Corollary 5.** *A ring $R$ is Artin semisimple if and only if every $R$-module is $\aleph_0$-injective.*

**Proof.** Let every $R$-module be $\aleph_0$-injective, then by Corollary 3, $R$ is regular and by the previous theorem $R$ is right Noetherian, i.e., the proof is complete. The converse is evident.

The following result and its remark settles our second question.

**Proposition 6.** *Let $R$ be a regular V-ring and let the socle $S$ of $R$ be finitely generated, then $R$ is selfinjective ($\aleph_0$-selfinjective) if and only if $\frac{R}{S}$ is selfinjective ($\aleph_0$-selfinjective).*

**Proof.** Clearly, $S = eR$, where $e$ is an idempotent. Since $R$ is a V-ring, $S$ is injective as a right $R$-module. Now, the injectivity ($\aleph_0$-injectivity) of $R$ implies that $\frac{R}{S}$ is injective ($\aleph_0$-selfinjective by Proposition 1) as an $R$-module, for $R = S \oplus (1-e)R$. Thus, $\frac{R}{S}$ is selfinjective, by Lemma 2. Conversely, if $\frac{R}{S}$ is selfinjective ($\aleph_0$-selfinjective), then it is injective ($\aleph_0$-injective) as an $R$-module by Lemma 2. Now, $R = S \oplus (1-e)R$ implies that $R$ is selfinjective ($\aleph_0$-selfinjective).

**Remark.** If $R$ is regular and the socle of $R$ is nonfinitely generated, then by a well-known theorem of Osofsky, see [10], $\frac{R}{S}$ is never injective as an $R$-module, i.e., by Lemma 2, $\frac{R}{S}$ is never selfinjective. But we note that it is always $\aleph_0$-selfinjective and $\aleph_0$-injective as an $R$-module.

**Note.** The converse part of Corollary 3 is also true when $R$ is a non-commutative ring, for the same proof of Proposition 1 works.

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# ON THE EXISTENCE OF FAMILIES OF ELLIPTIC CURVES OF RANK FOUR WITH ALL TWO-TORSION POINTS

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## Abstract

By using our method, we construct families of elliptic curves of rank $\geq 4$ with torsion structure $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ such that they are associated to a Kummer surface defined by explicit equations.

## 1. Introduction

The purpose of this paper is to construct families of elliptic curves of Mordell-Weil rank at least 4, with torsion group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In [9], we reduced the problem of construction of elliptic curves of rank $n$ for a given positive integer $n$ to the investigation of rational points on a threefold $V_n$. One of the merits of our reduction is the fact that the structure of $V_n$ as an algebraic variety gives us a rough idea about how hard it is to find elliptic curves of rank $n$. For example, recognizing that the Kodaira dimension of $V_n$ is negative for $n \leq 7$, we succeeded in [9] to construct all the elliptic curves of rank $n$ for each $n \leq 7$. Thus our variety $V_n$ should play a certain role too when we try to

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construct elliptic curves of high rank with a given torsion structure. In the present paper we show that $V_n$ does play an essential role for the investigation of elliptic curves of a given rank with torsion structure $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Here arises naturally a 2-dimensional subvariety $W_n'$ of $V_n$ according to the condition that the corresponding elliptic curve has the torsion structure. We are aware that Kulesz constructs an elliptic curve of rank $\geq 5$ with torsion structure $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ in [3], hence ours do not improve the known lower bound. Our method, however, gives an efficient perspective for where to search such curves. Moreover, a rich geometry of the parameter variety $W_4'$ may justify our use of it.

In the present paper, extending the method employed in [10], we construct a universal family $W_n'$ of elliptic curves $\{E_s\}$ with rank $n$ such that $E_s(\mathbf{Q})$ contains all the two-torsion points for every $s$. When $n = 4$, we will show that the parameter space $W_4'$ of our family has a structure of elliptic surface over $\mathbf{P}^1$. Furthermore, we will see that $W_4'$ is a Kummer surface associated with the product of two elliptic curves and determine their defining equations quite explicitly. Here arises a remarkable fact that the two elliptic curves are mutually quadratic twists of the others, and this fact enables us to determine the Picard number of $W_4'$ over the field of definition. It follows that the Mordell-Weil rank of $W_4'$ as an elliptic surface is found to be zero, nevertheless we can show it has infinitely many rational points, which in turn provide us with infinitely many elliptic curves with the desired property.

This paper is organized as follows. In Section 2 we recall the method of construction in [10] which deals with the case of one rational two-torsion point. In Section 3 we give the definition of the parameter space $W_4'$, and show that it is a Kummer surface. In the last part of this section, we formulate our main theorem and give a proof of it. At the end of this paper, we give a remark in the case of rank $n$ ($n = 1, 2, 3$).

## 2. Construction of Families of Elliptic Curves of Rank $n$ with Nontrivial Two-torsion Point

In this section we recall some results in [10] which will be needed to

construct our family. Let $E$ be an elliptic curve over a number field $k$ defined by the following equation:

$$E : y^2 = ax^3 + bx^2 + cx + d, \tag{1}$$

and let $f(x)$ denote the right hand side of (1). For an integer $n \geq 2$, let $V_n$ be the variety defined by the equation

$$V_n : z_i^2 = f(x_1)\, f(x_{i+1}), \;\; i = 1, \dots, n-1. \tag{2}$$

Since $V_n$ is birational to the quotient variety of $E^n$ by the action of $(-id, \dots, -id)$, the degree of the field extension $k(E^n)/k(V_n)$ is equal to 2.

The following result is fundamental:

**Theorem 2.1** [1, Section 4]. *Let $E_{f(x_1)}$ denote the quadratic twist of $E$ associated to the extension $k(E^n)/k(V_n)$. Then its defining equation is*

$$E_{f(x_1)} : f(x_1)y^2 = f(x),$$

*and the rank of $E_{f(x_1)}(k(V_n))$ is just $n$. Its generators are given by $(x_1, 1)$, $(x_{i+1}, z_i/f(x_1))$, $i = 1, \dots, n-1$.*

By Theorem 3.1 in [9] any given elliptic curve of rank $n$ is obtained by specializing generic elliptic curve at a certain rational point on $V_n$. This means rational points on $V_n$ parametrize elliptic curves of rank $\geq n$. Put $d = 0$ in the equation (1), then $(0, 0)$ is a two-torsion point on $E_{f(x_1)}$. By a similar argument to the proof of Theorem 3.1 in [9], the rational points on $V_n$ with $d = 0$ also parametrize ones of rank $\geq n$ with one nontrivial two-torsion point. Consequently, all such elliptic curves are found out to be obtained as a specialization at a certain point on $V_n$ with $d = 0$.

From now on, we focus on the variety $V_n$ associated to $f(x) = ax^3 + bx^2 + cx$. In order to investigate the rational points on $V_n$, we put $x_i = \alpha_{i-1}$, $i = 1, \dots, n$, and regard $\alpha_{i-1}$ as independent variables and $a$, $b$,

$c$, $z_i$, $i = 1, \ldots, n-1$, as variables. We denote $k(\sqrt{\alpha_0}, \ldots, \sqrt{\alpha_{n-1}})$ by $K$ and regard $V_n$ as a variety over $K$. Then $V_n$ is a surface in the projective space $\mathbf{P}^{n+1}$. We will show that it has a nonsingular model $W_n$ over $K$, by constructing a $K$-birational map of $V_n$ to $W_n$. For ease of description we introduce some notation.

**Notation 2.2.** We denote by $\begin{vmatrix} 0 & 1 & \cdots & n-1 \\ Z_0^2 & Z_1^2 & \cdots & Z_{n-1}^2 \end{vmatrix}$ the $n \times n$ determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_0 & \alpha_1 & \cdots & \alpha_{n-1} \\ \alpha_0^2 & \alpha_1^2 & \cdots & \alpha_{n-1}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_0^{n-2} & \alpha_1^{n-2} & \cdots & \alpha_{n-1}^{n-2} \\ Z_0^2 & Z_1^2 & \cdots & Z_{n-1}^2 \end{vmatrix}.$$

Using this notation, the birational model can be expressed in a simple form as follows:

**Theorem 2.3.** *If $n \geq 4$, then $V_n$ is $K$-birational to the variety $W_n$ defined by the equations,*

$$\begin{vmatrix} 0 & 1 & 2 & l+2 \\ Z_0^2 & Z_1^2 & Z_2^2 & Z_{l+2}^2 \end{vmatrix} = 0, \quad l = 1, \ldots, n-3. \tag{3}$$

**Proof.** One can check that the map $\varphi : V_n \to W_n$ defined by

$$(a, b, c, z_1, \ldots, z_{n-1}) \mapsto ((a\alpha_0^3 + b\alpha_0^2 + c\alpha_0)/\sqrt{\alpha_0}, z_1/\sqrt{\alpha_1}, \ldots, z_{n-1}/\sqrt{\alpha_{n-1}})$$

is birational, since the map defined by

$$(Z_0, \ldots, Z_{n-1}) \mapsto \left( \begin{vmatrix} 1 & 1 & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix}, - \begin{vmatrix} 1 & 1 & 1 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix}, \right.$$

$$\left.\begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix}, \sqrt{\alpha_0\alpha_1}Z_0Z_1X, \ldots, \sqrt{\alpha_0\alpha_{n-1}}Z_0Z_{n-1}X \right),$$

where

$$X = \begin{vmatrix} 1 & 1 & 1 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 \\ \alpha_0^4 & \alpha_1^4 & \alpha_2^4 \end{vmatrix},$$

gives the inverse map.

It is shown in [2] that $W_n$ is a nonsingular $(\underbrace{2, \ldots, 2}_{(n-3)\text{ times}})$ - complete intersection in $\mathbf{P}^{n-1}$. Moreover, it follows from [2] that $W_n$ contains $2^{n-1}$ lines defined by the equation:

$$Z_0 = s + t\alpha_0,\ Z_i = (-1)^{\varepsilon_i}(s + t\alpha_i),\ \varepsilon_i = 0 \text{ or } 1,\ i = 1, \ldots, n-1. \tag{4}$$

We denote the line (4) by $\ell(\varepsilon_1 \cdots \varepsilon_{n-1})$.

**Theorem 2.4** [2, Proposition 3.3]. *Let* $\varepsilon = (\varepsilon_1 \cdots \varepsilon_{n-1})$ *and* $\varepsilon' = (\varepsilon_1' \cdots \varepsilon_{n-1}')$. *Then* $\ell(\varepsilon)$ *and* $\ell(\varepsilon')$ *intersect if and only if*

$$\#\{i \mid \varepsilon_i \neq \varepsilon_i',\ i = 1, \ldots, n-1\} = 1 \ \text{ or } \ n-1.$$

**Remark.** The defining equation of the elliptic curve which corresponds to a point $(Z_0, \ldots, Z_{n-1})$ on $W_n$, is expressed as

$$\begin{vmatrix} 1 & 1 & 1 & x \\ \alpha_0 & \alpha_1 & \alpha_2 & x^2 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 & x^3 \\ Z_0^2 & Z_1^2 & Z_2^2 & y^2 \end{vmatrix} = 0,$$

namely,

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 \end{vmatrix} y^2 = \begin{vmatrix} 1 & 1 & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix} x^3 - \begin{vmatrix} 1 & 1 & 1 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix} x^2 + \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix} x. \tag{5}$$

This follows directly from the definition of the birational map $\varphi$. (5) defines a family of elliptic curves $\pi : \mathbf{E} \to W_n$.

## 3. Construction of Elliptic Curves of Rank Four with all Two-torsion Points Rational

### 3.1. Definition of the surface $W_4'$

Note that all of four two-torsion points on (5) are $K$-rational if and only if the right hand side of (5) is factored into linear factors over $K$. This means that the discriminant of the right hand side of (5) is a square in $K^*$. Therefore, we define the surface $W_4'$ by adding the following equation to those for $W_4$:

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix}^2 - 4 \begin{vmatrix} 1 & 1 & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix} \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix} = w^2, \tag{6}$$

and we denote by $\pi' : \mathbf{E}' \to W_4'$ the family of elliptic curves obtained from pulling back $\pi : \mathbf{E} \to W_4$ along the natural projection $W_4' \to W_4$. We focus our attention on the rational points on $W_4'$. From now on we denote $\alpha_i - \alpha_j$ by $(i\ j)$.

### 3.2. The structure of the surface $W_4'$

In this subsection, we obtain the following theorem:

**Theorem 3.1.** *Let $E_1/K$ and $E_2/K$ be the elliptic curves defined by*

$$E_1 : z_1^2 = x((2\ 0)(3\ 1)x + (3\ 2)^2)((1\ 0)x + (3\ 2)), \tag{7}$$

$$E_2 : z_2^2 = y((1\ 0)(2\ 0)y + 1)((2\ 0)(3\ 0)y + 1)((1\ 0)(3\ 0)y + 1), \tag{8}$$

*and let $g_1(x)$ (resp. $g_2(y)$) be the right hand side of (7) (resp. (8)). Then $W_4'$ is K-birational to the Kummer surface associated to the abelian surface $E_1 \times E_2$, and is defined by the following equation:*

$$S_1 : z^2 = g_1(x)\, g_2(y). \tag{9}$$

*The values of j-invariants of $E_1$ and $E_2$ are one and the same. It is given by*

$$\frac{2^8((2\ 1)^2(3\ 0)^2 + (1\ 0)(2\ 0)(3\ 1)(3\ 2))^3}{(1\ 0)^2(2\ 0)^2(2\ 1)^2(3\ 0)^2(3\ 1)^2(3\ 2)^2}.$$

*The smallest field over which $E_1$ and $E_2$ become isomorphic is $K(\sqrt{(3\ 2)})$. Moreover, the Picard number $\rho_K(W_4')$ of $W_4'$ over $K$ is 18.*

In order to prove this, we need some Lemmas.

**Lemma 3.1.1.** *$W_4'$ is K-birational to the surface $S_2$ defined by following equation:*

$$\begin{aligned} S_2 : u^2 = {}& ((3\ 2)s - (3\ 0))((3\ 2)t - (3\ 1))((3\ 1)s - (3\ 0)t) \\ & \cdot ((3\ 1)s + (2\ 0)t - (3\ 1))((2\ 1)s + (3\ 0)t - (3\ 0)) \\ & \cdot ((3\ 2)s - (3\ 2)t + (1\ 0))((2\ 1)s - (2\ 0)t + (1\ 0)) \\ & \cdot ((2\ 1)(3\ 1)(3\ 2)s - (2\ 0)(3\ 0)(3\ 2)t + (1\ 0)(3\ 0)(3\ 1)). \end{aligned} \tag{10}$$

*Proof.* Setting $Z_3 = 1$ in (3) we consider it in the affine space $\mathbf{A}^3$. Then $W_4$ is a quadratic surface in $\mathbf{A}^3$ with a rational point

$(Z_0, Z_1, Z_2) = (1, 1, 1)$. Therefore, one can easily parametrize all points on this surface as follows:

$$(Z_0, Z_1, Z_2) = (sT + 1, tT + 1, T + 1),$$

where

$$T = -2\begin{vmatrix} 0 & 1 & 2 & 3 \\ s & t & 1 & 0 \end{vmatrix}\begin{vmatrix} 0 & 1 & 2 & 3 \\ s^2 & t^2 & 1 & 0 \end{vmatrix}^{-1}$$

Insert this into (6) and put

$$u = 2^{-2}(1\ 0)^{-1}(2\ 0)^{-1}(2\ 1)^{-1}\begin{vmatrix} 0 & 1 & 2 & 3 \\ s^2 & t^2 & 1 & 0 \end{vmatrix}^2,$$

then we obtain the equation in (10).

**Lemma 3.1.2.** *$S_2$ is K-birational to $S_1$ defined by* (9).

**Proof.** When we regard $S_2$ as a double covering of $\mathbf{A}^2$ with coordinate $(s, t)$, the ramification places are lines which correspond to the linear factors of the right hand side of (10). We call them $\widetilde{\ell}_1, \ldots, \widetilde{\ell}_8$ respectively. Then lines $\widetilde{\ell}_1$, $\widetilde{\ell}_2$, $\widetilde{\ell}_3$, $\widetilde{\ell}_7$, $\widetilde{\ell}_8$ intersect at $(s, t) = ((3\ 0)(3\ 2)^{-1}, (3\ 1)(3\ 2)^{-1})$. In order to translate this point to the origin $(0, 0)$, we let

$$s = x + (3\ 0)(3\ 2)^{-1}, \quad t = y + (3\ 1)(3\ 2)^{-1}.$$

Further, we put

$$u = (3\ 2)^3 w.$$

Then the equation (10) becomes

$$\begin{aligned} w^2 = {} & xy((2\ 1)x - (2\ 0)y)((3\ 1)x - (3\ 0)y) \\ & \cdot ((2\ 1)(3\ 1)x - (2\ 0)(3\ 0)y)((3\ 2)x - (3\ 2)y + 2(1\ 0)) \\ & \cdot ((3\ 1)(3\ 2)x + (2\ 0)(3\ 2)y + 2(2\ 0)(3\ 1)) \\ & \cdot ((2\ 1)(3\ 2)x + (3\ 0)(3\ 2)y + 2(2\ 1)(3\ 0)). \end{aligned} \tag{11}$$

Next, we regard this surface as double covering of $\mathbf{A}^2$ with coordinate $(x, y)$, and call the ramification lines $\widetilde{m}_1, \ldots, \widetilde{m}_8$, respectively. Then $\widetilde{m}_5, \widetilde{m}_6, \widetilde{m}_7, \widetilde{m}_8$ intersect at

$$(x, y) = \left(-\frac{2(2\ 0)(3\ 0)}{(3\ 2)((2\ 0)+(3\ 1))}, -\frac{2(2\ 1)(3\ 1)}{(3\ 2)((2\ 0)+(3\ 1))}\right).$$

In order to map this point to infinity, homogenize (11) by a variable $z$, and put

$$x = \frac{1}{(2\ 1)(3\ 1)}(X + (20)(30)Z),$$

$$y = Z,$$

$$z = -\frac{1}{2(2\ 1)}\left(\frac{(3\ 2)}{(3\ 0)(3\ 1)}X - \frac{1}{(3\ 0)}Y + \frac{(3\ 2)}{(3\ 1)}((2\ 0)+(3\ 1))Z\right),$$

$$w = \frac{W}{(2\ 1)^2(3\ 0)(3\ 1)}.$$

Then we obtain

$$\begin{aligned} W^2 &= XYZ(X + (1\ 0)(2\ 0)Z)(X + (2\ 0)(3\ 0)Z) \\ &\quad \cdot (X + (1\ 0)(3\ 0)Z)((3\ 2)X + (1\ 0)Y) \\ &\quad \cdot ((3\ 2)^2X + (2\ 0)(3\ 1)Y). \end{aligned}$$

Put $X = 1$ and rename the variables $Y$ to $x$, $Z$ to $y$, and $W$ to $z$, then we obtain the equation (9).

**Proof of Theorem 3.1.** $W_4'$ is $K$-birational to $S_1$ by Lemmas 3.1.1 and 3.1.2. Next, we show that the smallest field over which $E_1$ is isomorphic to $E_2$ is $K\sqrt{(3\ 2)}$. Translate $E_1$ and $E_2$ to canonical forms as follows. As for $E_1$, multiply the both hand sides of (7) by $(1\ 0)^2(2\ 0)^2(3\ 1)^2$, and put $x' = (1\ 0)(2\ 0)(3\ 1)x$, $z_1' = (1\ 0)(2\ 0)(3\ 1)z_1$. Further, put $x' = x'' - (3\ 2)((1\ 0)(3\ 2) + (2\ 0)(3\ 1))/3$ to eliminate the degree two term in $x'$. Then the equation for $E_1$ becomes

$$z_1'^2 = x''^3 + A_1 x'' + B_1, \tag{12}$$

where

$$A_1 = -(3\ 2)^2((1\ 0)^2(3\ 2)^2 + (2\ 0)^2(3\ 1)^2 - (1\ 0)(2\ 0)(3\ 1)(3\ 2))/3,$$

$$B_1 = (3\ 2)^3((1\ 0)(3\ 2) + (2\ 0)(3\ 1))(2(2\ 1)^2(3\ 0)^2 - (1\ 0)(2\ 0)(3\ 1)(3\ 2))/27.$$

On the other hand, as for $E_2$, put $y' = 1/y$, multiply the both hand sides of (8) by $y'^4$, and put $z_2' = y'^2 z_2$. Then the equation for $E_2$ becomes

$$\begin{aligned} z_2'^2 = y'^3 &+ ((1\ 0)(2\ 0) + (2\ 0)(3\ 0) + (3\ 0)(1\ 0))\,y'^2 \\ &+ (1\ 0)(2\ 0)(3\ 0)((1\ 0) + (2\ 0) + (3\ 0))\,y' \\ &+ (1\ 0)^2(2\ 0)^2(3\ 0)^2. \end{aligned} \tag{13}$$

Further, put

$$y'' = y' - ((1\ 0)(2\ 0) + (2\ 0)(3\ 0) + (3\ 0)(1\ 0))/3$$

to eliminate degree two term of $y'$. Then (13) becomes

$$z_2'^2 = y''^3 + \frac{A_1}{(3\ 2)^2}\,y'' + \frac{B_2}{(3\ 2)^3}. \tag{14}$$

It follows from (12) and (14) that

$$j(E_1) = j(E_2) = \frac{2^8((2\ 1)^2(3\ 0)^2 + (1\ 0)(2\ 0)(3\ 1)(3\ 2))^3}{(1\ 0)^2(2\ 0)^2(2\ 1)^2(3\ 0)^2(3\ 1)^2(3\ 2)^2} \tag{15}$$

and that the smallest field over which $E_1$ and $E_2$ are isomorphic is $K\sqrt{(3\ 2)}$. Both $E_1$ and $E_2$ have no complex multiplication, since $j(E_1) = j(E_2)$ is transcendental. Therefore, the Picard number $\rho_K(W_4')$ is 18.

### 3.3. On the structure of $W_4'$ as elliptic surface

The purpose of this subsection is to show that $W_4'$ has a structure of elliptic surface over $\mathbf{P}^1$, and to compute the rank of the group of its sections.

**Theorem 3.2.** *$W_4'$ can be given a structure of elliptic surface over the projective line $p : W_4' \to \mathbf{P}^1$. Its defining equation is given by*

$$
\begin{aligned}
y^2 = {} & \lambda\mu((3\ 0)\lambda - (2\ 0)\mu)((3\ 1)\lambda - (2\ 1)\mu) \\
& \cdot (\mu x - \lambda)((3\ 0)(3\ 1)\lambda x - (2\ 0)(2\ 1)\mu) \\
& \cdot ((3\ 0)((3\ 1)\lambda - (2\ 1)\mu)x + (2\ 1)((3\ 0)\lambda - (2\ 0)\mu)) \\
& \cdot ((3\ 1)((3\ 0)\lambda - (2\ 0)\mu)x + (2\ 0)((3\ 1)\lambda - (2\ 1)\mu)),
\end{aligned} \tag{16}
$$

*where $(\lambda : \mu)$ denotes the projective coordinate of $\mathbf{P}^1$.*

**Proof.** As is noted in [8], the left hand side of (3) which is the defining equation for $W_4$, is expressed as

$$
\begin{aligned}
\text{(the left hand side of (3))} = {} & (0\ 3)\begin{vmatrix} 0 & 1 & 2 \\ Z_0 & Z_1 & -Z_2 \end{vmatrix}\begin{vmatrix} 1 & 2 & 3 \\ Z_1 & Z_2 & -Z_3 \end{vmatrix} \\
& + (1\ 2)\begin{vmatrix} 0 & 1 & 3 \\ Z_0 & Z_1 & -Z_3 \end{vmatrix}\begin{vmatrix} 0 & 2 & 3 \\ Z_0 & -Z_2 & Z_3 \end{vmatrix}.
\end{aligned}
$$

Putting

$$
\lambda\begin{vmatrix} 0 & 1 & 3 \\ Z_0 & Z_1 & -Z_3 \end{vmatrix} = \mu\begin{vmatrix} 0 & 1 & 2 \\ Z_0 & Z_1 & -Z_2 \end{vmatrix} \tag{17}
$$

and inserting this into (3) we get the following equation:

$$
(0\ 3)\lambda\begin{vmatrix} 1 & 2 & 3 \\ Z_1 & Z_2 & -Z_3 \end{vmatrix} = (2\ 1)\mu\begin{vmatrix} 0 & 2 & 3 \\ Z_0 & -Z_2 & Z_3 \end{vmatrix}. \tag{18}
$$

By (17) and (18), $Z_0$ and $Z_1$ are expressed by $Z_2$ and $Z_3$,

$$
Z_0 = ((3\ 0)AZ_2 + (2\ 0)BZ_3)/C, \tag{19}
$$

$$
Z_1 = ((3\ 1)BZ_2 + (2\ 1)AZ_3)/C, \tag{20}
$$

where

$$A = (3\ 0)(3\ 1)\lambda^2 - 2(2\ 0)(3\ 1)\lambda\mu + (2\ 0)(2\ 1)\mu^2, \tag{21}$$

$$B = (3\ 0)(3\ 1)\lambda^2 - 2(2\ 1)(3\ 0)\lambda\mu + (2\ 0)(2\ 1)\mu^2, \tag{22}$$

$$C = (3\ 2)((3\ 0)(3\ 1)\lambda^2 - (2\ 0)(2\ 1)\mu^2). \tag{23}$$

Note that the left hand side of (6)

$$= (2\ 1)^4 Z_0^4 + (2\ 0) Z_1^4 + (1\ 0)^4 Z_2^4$$

$$- 2((2\ 0)^2(2\ 1)^2 Z_0^2 Z_1^2 + (1\ 0)^2(2\ 0)^2 Z_1^2 Z_2^2 + (1\ 0)^2(2\ 1)^2 Z_0^2 Z_2^2)$$

$$= ((2\ 1) Z_0 + (2\ 0) Z_1 + (1\ 0) Z_2)((2\ 1) Z_0 + (2\ 0) Z_1 - (1\ 0) Z_2)$$

$$\cdot ((2\ 1) Z_0 + (2\ 0) Z_1 - (1\ 0) Z_2)((2\ 1) Z_0 - (2\ 0) Z_1 - (1\ 0) Z_2).$$

The latter equality comes from the following identity:

$$a^4x^4 + b^4y^4 + c^4z^4 - 2(a^2b^2x^2y^2 + b^2c^2y^2z^2 + a^2c^2x^2z^2)$$

$$= (ax + by + cz)(ax + by - cz)(ax - by + cz)(ax - by - cz). \tag{24}$$

Inserting (19), (20) into this, and letting $x = Z_2/Z_3$, $y' = C^2 w/Z_3$, we see that (6) becomes as follows:

$$y'^2 = 2^4(1\ 0)^2(2\ 0)^2(2\ 1)^2(3\ 2)^2\lambda\mu((3\ 0)\lambda - (2\ 0)\mu)((3\ 1)\lambda - (2\ 1)\mu)$$

$$\cdot (\mu x - \lambda)((3\ 0)(3\ 1)\lambda x - (2\ 0)(2\ 1)\mu)$$

$$\cdot ((3\ 0)((3\ 1)\lambda - (2\ 1)\mu)x + (2\ 1)((3\ 0)\lambda - (2\ 0)\mu))$$

$$\cdot ((3\ 1)((3\ 0)\lambda - (2\ 0)\mu)x + (2\ 0)((3\ 1)\lambda - (2\ 1)\mu)).$$

Divide both hand sides by $2^4(1\ 0)^2(2\ 0)^2(2\ 1)^2(3\ 2)^2$ and put $y = y'/2^2(1\ 0)(2\ 0)(2\ 1)(3\ 2)$. Then we obtain the equation (16).

We denote $W_4'$ by $W_{4e}'$ when it is regarded as elliptic surface over $P^1$.

**Theorem 3.3.** *The group of sections* $E_W(K)$ *of* $E_W$ *over* $K$ *consists of the four lines* $\ell(000)$, $\ell(101)$, $\ell(110)$, $\ell(011)$, *and is isomorphic to* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. *In particular, it has rank zero.*

**Proof.** We will see how the eight lines mentioned in Section 2 are situated in $E_W$. Note that the value of the determinant $\begin{vmatrix} 0 & 1 & 2 \\ Z_0 & Z_1 & -Z_2 \end{vmatrix}$ (resp. $\begin{vmatrix} 0 & 1 & 3 \\ Z_0 & Z_1 & -Z_3 \end{vmatrix}$) appearing in (17) is independent on $Z_3$ (resp. $Z_2$). Furthermore, their values on the eight lines $\ell(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $\varepsilon_i = 0, 1$, can be computed as follows:

$$\begin{vmatrix} 0 & 1 & 2 \\ Z_0 & Z_1 & -Z_2 \end{vmatrix} = \begin{cases} -2(1\ 0)(s + t\alpha_2), & \text{on } \ell(000), \ell(001), \\ 0, & \text{on } \ell(010), \ell(011), \\ 2(2\ 1)(s + t\alpha_0), & \text{on } \ell(100), \ell(101), \\ 2(2\ 0)(s + t\alpha_1), & \text{on } \ell(110), \ell(111), \end{cases}$$

$$\begin{vmatrix} 0 & 1 & 3 \\ Z_0 & Z_1 & -Z_3 \end{vmatrix} = \begin{cases} -2(1\ 0)(s + t\alpha_3), & \text{on } \ell(000), \ell(010), \\ 0, & \text{on } \ell(001), \ell(011), \\ 2(3\ 1)(s + t\alpha_0), & \text{on } \ell(100), \ell(110), \\ 2(3\ 0)(s + t\alpha_1), & \text{on } \ell(101), \ell(111). \end{cases}$$

Hence it follows from (18) that the values of the coordinate $(\lambda : \mu)$ on the lines are given by

$$(\lambda : \mu) = \begin{cases} (s + t\alpha_2 : s + t\alpha_3), & \text{on } \ell(000), \\ (1 : 0), & \text{on } \ell(001), \\ (0 : 1), & \text{on } \ell(010), \\ ((2\ 0)(2\ 1)(s + t\alpha_3) : (3\ 0)(3\ 2)(s + t\alpha_0)), & \text{on } \ell(011), \\ ((2\ 1) : (3\ 1)), & \text{on } \ell(100), \\ ((2\ 1)(s + t\alpha_0) : (3\ 0)(s + t\alpha_1)), & \text{on } \ell(101), \\ ((2\ 0)(s + t\alpha_1) : (3\ 1)(s + t\alpha_0)), & \text{on } \ell(110), \\ ((2\ 0) : (3\ 0)), & \text{on } \ell(111). \end{cases} \tag{25}$$

From this it is found out that each of $\ell(001)$, $\ell(010)$, $\ell(100)$, $\ell(111)$ is contained in a fiber and that $\ell(000)$, $\ell(101)$, $\ell(110)$, $\ell(011)$ are sections. Recall that the defining equation of $\ell(000)$ is

$$\begin{vmatrix} 0 & 1 & 2 \\ Z_0 & Z_1 & Z_2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 3 \\ Z_0 & Z_1 & Z_3 \end{vmatrix} = 0$$

(see (4)). Eliminating $Z_0$, we have

$$-(1\ 0)(3\ 2)Z_1 + (1\ 0)(3\ 1)Z_2 - (1\ 0)(2\ 1)Z_3 = 0.$$

Inserting (20) into this and dividing the both hand sides by $2(1\ 0)(2\ 1)(3\ 1)$, we obtain

$$((3\ 0)\lambda - (2\ 0))(Z_2 - \lambda Z_3) = 0.$$

Since $x = Z_2/Z_3$ this equation implies that the line $\ell(000)$ corresponds to the section defined by $x = \lambda$. By a similar argument, we see that the equations of the other three lines are as follows:

$$x = \frac{(2\ 0)(2\ 1)}{(3\ 0)(3\ 1)\lambda} \qquad \text{for } \ell(011),$$

$$x = -\frac{(2\ 1)((3\ 0)\lambda - (2\ 0))}{(3\ 0)((3\ 1)\lambda - (2\ 1))} \quad \text{for } \ell(101),$$

$$x = -\frac{(2\ 0)((3\ 1)\lambda - (2\ 1))}{(3\ 1)((3\ 0)\lambda - (2\ 0))} \quad \text{for } \ell(110).$$

Consequently, by the equation (16), these four lines give all the two-torsion sections of $E_W$.

Next, we investigate what kind of singular fibers appear on $E_W$. As is mentioned above, it has four disjoint lines each of which must appear as an irreducible component of some singular fiber $F_i$, $i = 1, \ldots, 4$. After resolving the singularities of $W_4'$, the Euler characteristic $e(F_i)$ must be $\geq 6$. On the other hand, since $W_4'$ is a Kummer surface, the Euler characteristic of the minimal nonsingular model of $W_4'$ is equal to 24.

Therefore, we have $e(F_i) = 6$, $i = 1, \ldots, 4$, and it follows that there is no other singular fiber. Hence each singular fiber is of type $I_0^*$ in Kodaira's symbol (cf. [6]). Since the group $C(K)/C_0(K)$ of its irreducible components is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, and each of the four sections $\ell(000)$, $\ell(101)$, $\ell(110)$, $\ell(011)$ intersects with mutually different irreducible component, we have $E_W(K)_{tor} \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Let $r_K$ denote the rank of the group $E_W(K(\mathbf{P}^1))$ of $K$-rational sections of the elliptic surface $E_W$. Since all the irreducible components of the singular fibers are $K$-rational, and $\rho_K = 18$, by Theorem 3.1, it follows from the formula [5]

$$\rho = r + 2 + \sum_{(\lambda,\, \mu): E_{(\lambda,\, \mu)} \text{ is singular}} (\#\,(\text{irreducible component of } E_{(\lambda,\, \mu)}) - 1)$$

that $r_K = 0$.

Therefore, there are not enough sections over $K$ to produce infinitely many $K$-rational points on $E_W$. But, we can still find a four-fold section over $K$.

**Lemma 3.3.1.** *The elliptic curve $E_4$ defined by the equation*

$$E_4 : \begin{vmatrix} 0 & 1 & 2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix} = 0, \begin{vmatrix} 0 & 1 & 3 \\ Z_0^2 & Z_1^2 & Z_3^2 \end{vmatrix} = 0 \tag{26}$$

*in $\mathbf{P}^3$ is contained in $W_4'$ as a four-fold section.*

*Proof.* Let $P = (Z_0, Z_1, Z_2, Z_3)$ be an arbitrary point on $E_4$. Then (26) implies

$$\operatorname{rank}\begin{pmatrix} 1 & 1 & 1 & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ Z_0^2 & Z_1^2 & Z_2^2 & Z_3^2 \end{pmatrix} < 3,$$

hence $P \in W_4$. Put

$$w = \begin{vmatrix} 1 & 1 & 1 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix}.$$

Then the point $(Z_0, Z_1, Z_2, Z_3, w)$ satisfies (6). Therefore, $E_4$ lies in $W_4'$. In order to show that $E_4$ is a four-fold cover of $\mathbf{P}^1$, we may specialize the $\alpha_i$. Put $\alpha_i = i+1$, $i = 0, \ldots, 3$, $\lambda = \mu = 1$, and $Z_3 = 1$. Then by (17), we have

$$Z_0 = Z_1 - Z_2 + 1. \tag{27}$$

Insert (27) into the first equation of (26). Then $Z_1$ becomes

$$Z_1 = -Z_2 + 1 \pm \sqrt{3Z_2^2 - 4Z_2 + 2}. \tag{28}$$

By inserting (27), (28) into the second equation of (26), we see that it becomes an equation of degree 4. This implies that $E_4$ is a four-fold cover of $\mathbf{P}^1$.

The next lemma shows that $E_4$ has infinitely many rational points.

**Lemma 3.3.2.** *Let* $K' = k(\alpha_0, \ldots, \alpha_3)$. *Then the Mordell-Weil group* $E_4(K')$ *is infinite.*

**Proof.** Note that $E_4$ has a $K'$-rational point $(Z_0, \ldots, Z_3) = (1, 1, 1, 1)$. It is sufficient to show this by specialization. Put $\alpha_0 = 1$, $\alpha_1 = 4$, $\alpha_2 = 9$, $\alpha_3 = 16$. $E_4$ is isogenous to the elliptic curve $E_4'$ defined by the equation

$$E_4' : y^2 = x(8x-5)(15x-12), \tag{29}$$

by the map

$$\psi : (Z_0, Z_1, Z_2, Z_3) \mapsto (Z_1^2/Z_0^2,\ 3Z_1Z_2Z_3/Z_0^3).$$

Under this isogeny, the rational point $(1, 1, 1, 1)$ corresponds to $(1, 3)$.

Transform the coefficient of the term in $x^3$ of (29) to 1, and eliminate the term in $x^2$, then $E_4'$ becomes

$$E_4'' : y^2 = x^3 - 2547x + 40014,$$

and $(1, 3) \in E_4'(k)$ goes to $(63, 360) \in E_4''(k)$. Since one can check that this rational point is not torsion, the statement of the lemma is proved.

Let $E_{(\lambda, \mu)}$ denote the fiber of the elliptic surface $p : E_W \to \mathbf{P}^1$ at $(\lambda, \mu) \in \mathbf{P}^1$.

**Lemma 3.3.3.** *There exist infinitely many K-rational points* $Q_i \in E_4(K)$, $i = 1, 2, \ldots$, *such that*

(i) $Q_i \in E_{(\lambda_i, \mu_i)}$ *with* $\lambda_i, \mu_i \in K$,

(ii) $E_{(\lambda_i, \mu_i)}$ *is a nonsingular fiber of* $p$,

(iii) $Q_i$ *is not torsion on* $E_{(\lambda_i, \mu_i)}$.

**Proof.** For, by Merel's theorem [4], there are at most finite torsion points on each special fiber $E_{(\lambda, \mu)}$, $(\lambda, \mu) \in \mathbf{P}^1(K)$, hence the number of torsion sections is also finite. Therefore, they intersect $E_4$ in at most finitely many points. Moreover, the values of $\lambda$, $\mu$ which give rational points on $E_4$ belong to $K$ by (17), hence there are infinitely many rational points $Q_i \in E_4(K)$ with the desired properties.

## 3.4. Main result

Now, we will prove the main theorem of this article.

**Theorem 3.4.** *There are infinitely many elliptic surfaces* $\pi_i : \mathbf{E}_i \to C_i$, $i = 1, 2, \ldots$ *such that, for any* $i \geq 1$,

(i) $\mathbf{E}_i$, $C_i$, *and* $\pi_i$ *are defined over K*,

(ii) $\pi_i : \mathbf{E}_i \to C_i$ *is not isotrivial, namely the j-invariant of its fibers is nonconstant*,

(iii) *$C_i$ is an elliptic curve with infinitely many $K$-rational points $P_i^n$, $n = 1, 2, \ldots,$*

(iv) *the fiber $\pi_i^{-1}(P_i^n)$ is an elliptic curve of $K$-rank $\geq 4$ with four $K$-rational two-torsion points.*

**Proof.** Let $C_i = E_{(\lambda_i, \mu_i)}$ in the notation of Lemma 3.3.3 and let $\pi_i : \mathbf{E}_i \to C_i$ be the restriction of $\pi' : \mathbf{E}' \to W_4'$ to $C_i \subset W_4'$. It follows from Silverman's theorem [7] that rational point on $C_i$ such that the specialization map for $\pi_i : \mathbf{E}_i \to C_i$ is not injective has bounded height. Since we know by Lemma 3.3.3 that $Q_i \in C_i(K)$ is of infinite order, this implies that there is an integer $n_0(\geq 1)$ such that for any integer $n \geq n_0$, $nQ_i$ gives injective specialization map. Letting $P_i^n = (n + n_0)Q_i$, $n = 1, 2, \ldots,$ we see that the properties (i), (iii), (iv) hold true. Finally, we show that these families $\pi_i : \mathbf{E}_i \to C_i$ satisfy the property (ii). Since $\pi' : \mathbf{E}' \to W_4'$ is a pull-back of $\pi : \mathbf{E} \to W_4$, we have only to check that the $j$-invariant map of the fibers of $\pi|_{\alpha(C_i)}$, where $\alpha$ denotes the natural projection map $W_4' \to W_4$, is nonconstant. By [10, Lemma 3.1], the value of $j$-invariant of elliptic curve which corresponds to a rational point on $W_4$ is

$$2^8\left(\begin{vmatrix} 1 & 1 & 1 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix}^2 - 3\begin{vmatrix} 1 & 1 & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix}\begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix}\right)^3$$

$$\cdot\left(\begin{vmatrix} 1 & 1 & 1 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix}^2 - 4\begin{vmatrix} 1 & 1 & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix}\begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix}\right)^{-1}$$

$$\cdot \begin{vmatrix} 1 & 1 & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix}^{-2} \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 \\ Z_0^2 & Z_1^2 & Z_2^2 \end{vmatrix}^{-2} . \tag{30}$$

Insert (19), (20) for $Z_0$, $Z_1$, and put $\alpha_i = (i+1)^2$, if $j$-invariant is 0, then the numerator of (30) is zero, that is

$$\begin{aligned}
&16128000000000000 \\
&\cdot((911042990838864\,\lambda^8 - 3093525301404672\,\lambda^7 + 4009641123295872\,\lambda^6 \\
&- 2672723011718304\,\lambda^5 + 1016863438519755\,\lambda^4 - 227094112106784\,\lambda^3 \\
&+ 28947385613952\,\lambda^2 - 1897620820992\lambda + 47483919184)\,x^4 \\
&+ 2(773381325351168\,\lambda^8 - 2159638324797024\,\lambda^7 \\
&+ 2464527230076024\,\lambda^6 - 1527896130704523\,\lambda^5 \\
&+ 567735280266960\,\lambda^4 - 129821239863783\,\lambda^3 \\
&+ 17792520051384\,\lambda^2 - 1324758730464\lambda + 40308939008)\,x^3 \\
&+ 7(110348086517532\,\lambda^8 - 288945185758416\,\lambda^7 + 314270321845491\,\lambda^6 \\
&- 189244229973552\,\lambda^5 + 69640679899440\,\lambda^4 - 16079575095792\,\lambda^3 \\
&+ 2268857464731\,\lambda^2 - 177243871376\lambda + 5751385692)\,x^2 \\
&+ 2(729954904794448\,\lambda^8 - 192013776222309\,\lambda^7 + 211879661964984\,\lambda^6 \\
&- 129097735057278\,\lambda^5 + 47713127772960\,\lambda^4 - 10969088599638\,\lambda^3 \\
&+ 1529653674744\,\lambda^2 - 117784502849\lambda + 3804553688)x \\
&+ 9274035199644\,\lambda^8 - 24808924868832\,\lambda^7 + 29477832362082\,\lambda^6 \\
&- 19085251109184\,\lambda^5 + 7212109965255\,\lambda^4 - 1621622643264\,\lambda^3 \\
&+ 212813604562\,\lambda^2 - 15218214752\lambda + 483366364) = 0.
\end{aligned} \tag{31}$$

This shows that the set of points which give elliptic curves of $j$-invariant 0 is a four-fold section of the elliptic surface, consequently $\pi_i$ is nonconstant.

**Remark.** For completeness, we describe briefly the cases of rank $n$ $(n = 1, 2, 3)$. Let $K' = k(\alpha_0, \ldots, \alpha_{n-1})$. In the case of rank 1, start with

$$f(\alpha_0)\, y^2 = f(x)$$

which has a free rational point $(\alpha_0, 1)$ and a torsion point $(0, 0)$. Consequently, $V_1$ is just $\mathbf{P}^2$ with coordinate $(a, b, c) \in \mathbf{P}^2$, on the other hand the equation of $W_1'$ is

$$b^2 - 4ac = w^2.$$

This is a hypersurface in $\mathbf{P}^3$ of degree 2 with rational point $(a, b, c, w) = (0, 0, 1, 0)$, hence is rational over $K'$. In the case of rank 2 and 3, by a similar argument we find that the defining equation of $W_n'$, $n = 2, 3$, is the following simultaneous equation

$$\begin{cases} z_j^2 = f(\alpha_0) f(\alpha_j), & j = 1, \ldots, n-1, \\ b^2 - 4ac = w^2. \end{cases}$$

Hence $W_2'$ is rational over $K'$ and so is $W_3'$ over $K$.

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# CONFORMAL CHANGES OF $(G, L)$-STRUCTURES IN A RIZZA STRUCTURE

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## Abstract

We consider the conformal change of Rizza manifold admitting a generalized Finsler metric $G_{ij}$ and a new conformal invariant non-linear connection $L^i_j$ constructed from the generalized Chern's non-linear connection $N^i_j$. Then we obtain the invariant tensors and conformally invariant $h$-Finsler connection under conformal changes, and find the condition for a $(G, L)$-structure of Rizza manifold to be a conformal flatness.


## 1. Introduction

Let $M^{2n}$ be a $2n$-dimensional Finsler manifold whose fundamental function is given by $L(x, y)$. The metric tensor $g_{ij}(x, y)$ is introduced by $g_{ij}(x, y) = \dot{\partial}_i \dot{\partial}_j L^2/2$. Moreover, $M^{2n}$ admits an almost complex structure $f^i_j(x)$. If $L$ satisfies Rizza condition $L(x, cy) = |\,\widetilde{c}\,| L(x, y)$ for any complex number $c$, then the Finsler manifold is called a *Rizza manifold* and the structure $(f^i_j(x), g_{ij}(x, y))$ is called a *Rizza structure*. Here, we can

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rewrite Rizza condition as $L(x, \phi_\theta y) = L(x, y)$ for any $\theta$ where we put $\phi^i_{\theta j} = \cos\theta\delta^i_j + \sin\theta f^i_j(x)$. The notion of the Rizza structure was, for the first time, introduced by G. B. Rizza. The Rizza manifold has been studied by many authors ([3], [7], [8], [13], [14], [15]).

In the present paper, first we consider the conformal theory of Rizza manifold admitting a generalized Finsler metric $G_{ij}$ and a new conformal invariant non-linear connection $L^i_j$ constructed from the generalized Chern's non-linear connection $N^i_j$, and we introduce a Finsler connection $F^*\Gamma$ with respect to $L^i_j$. Next, we find the invariant tensors under conformal changes of the above Finsler connection in Rizza manifold, and conformally invariant $h$-Finsler connections are introduced. Finally, we investigate the condition for a $(G, L)$-structure of Rizza manifold to be a conformally flat Finsler manifold.

## 2. Preliminaries

Let $M^{2n}$ be a $2n$-dimensional Rizza manifold with a Rizza structure $(f^i_j(x), g_{ij}(x, y))$. If we put

$$G_{ij}(x, y) = \frac{1}{2}(g_{ij}(x, y) + g_{pq}(x, y) f_i^p f_j^q), \tag{2.1}$$

then we see that $G_{ij} = G_{ji}$, $G_{ij}$ is (0) $p$-homogeneous for $y^i$ and $G_{ij}(x, y)\xi^i\xi^j$ is positive definite. That is, $G_{ij}$ is a generalized Finsler metric [8]. Moreover, we can see easily

$$G_{pq}(x, y) f_i^p(x) f_j^q(x) = G_{ij}(x, y). \tag{2.2}$$

It is known that the following identities in the Rizza manifold are as follows:

$$\begin{gathered} y^r\dot\partial_r G_{ij} = 0, \quad y^m f^r_m \dot\partial_r G_{ij} = 0, \\ \dot\partial_i G_{pq} y^p y^q = 0, \quad G_{ij} = G_{pq} f_i^p f_j^q. \end{gathered} \tag{2.3}$$

In the Rizza manifold, the generalized Chern's non-linear connection is given as follows [10]:

$$N_j^i = \frac{1}{2}(G^{ih}\dot{\partial}_j G_{hs} - f_h^i G^{hr} f_j^t \partial_t G_{rs} + S_{sj}^i - G^{ih} G_{ms} S_{jh}^m$$

$$- G^{ih}\dot{\partial}_r G_{hm} y^m S_{sj}^r + G^{ih} f_h^m G_{rs} f_t^r S_{mj}^t)\, y^s, \tag{2.4}$$

where $S_{kj}^i = (\partial f_r^i) f_j^r$. $N_j^i$ satisfies the law of coordinate transformation of a non-linear connection. Thus $N_j^i$ defined by (2.4) is a non-linear connection called *the generalized Chern's non-linear connection* in a Rizza manifold. It is known [10] that if the given almost complex structure $f_j^i(x)$ in the Rizza manifold is integrable, then $N_j^i$ defined by (2.4) is coincides with the Chern's non-linear connection. With respect to the generalized Chern's non-linear connection $N_j^i(x, y)$ and a generalized Finsler metric $G_{ij}(x, y)$ respectively defined by (2.1) and (2.4), we introduce a symmetric Finsler connection $F\Gamma = (\Gamma_{jk}^i, N_j^i, C_{jk}^i)$ as follows ([1], [11]):

$$\Gamma_{jk}^i = \frac{1}{2} G^{im}(X_j G_{mk} + X_k G_{mj} - X_m G_{jk}),$$

$$C_{jk}^i(x, y) = \frac{1}{2} G^{im}(\dot{\partial}_j G_{mk} + \dot{\partial}_k G_{mj} - \dot{\partial}_m G_{jk}), \tag{2.5}$$

where $X_j = \partial_j - N_j^m \dot{\partial}_m$, $\partial_j = \partial/\partial x^j$, $\dot{\partial}_j = \partial/\partial y^j$. Denoting the $h$-covariant, $v$-covariant derivative with respect to $F\Gamma$ by $\nabla$ and $\dot{\nabla}$ respectively, we have directly

$$\nabla_k G_{ij} = 0, \quad \dot{\nabla}_k G_{ij} = 0.$$

The above Finsler connection $F\Gamma = (\Gamma_{jk}^i, N_j^i, C_{jk}^i)$ is said to be the *Finsler connection* associated to a $(G, N)$-structure.

According to Matsumoto [11] we write the $h$-torsion and $hv$-torsion of $F\Gamma$ as

$$R^i_{jk} = X_k N^i_j - j/k, \quad P^i_{jk} = \dot{\partial}_k N^i_j - \Gamma^i_{jk}, \tag{2.6}$$

and the curvatures of $F\Gamma$ as

$$R^i_{hjk} = \{X_k \Gamma^i_{hj} + \Gamma^i_{mk}\Gamma^m_{hj} - j/k\} + C^i_{hm}R^m_{jk},$$

$$P^i_{hjk} = \dot{\partial}_k \Gamma^i_{hj} - \nabla_j C^i_{hk} + C^i_{hm}P^m_{jk}, \tag{2.7}$$

where $j/k$ denotes the interchange of indices $j$, $k$ of the preceding terms and we put

$$K^i_{hjk} = X_k \Gamma^i_{hj} + \Gamma^i_{mk}\Gamma^m_{hj} - j/k,$$

$$Q^i_{hjk} = \nabla_j C^i_{hk} - C^i_{hm}P^m_{jk}. \tag{2.8}$$

## 3. Conformal Invariant $h$-connection

In the Rizza manifold $M^{2n}$, let us consider conformal change:

$$G_{ij}(x, y) \to \overline{G}_{ij}(x, y) = e^{2\sigma(x)}G_{ij}(x, y), \tag{3.1}$$

where $\sigma(x)$ is any scalar. Then we can see easily

$$\overline{G}^{ij} = e^{-2\sigma(x)}G^{ij},$$

$$\partial_k \overline{G}_{ij} = e^{2\sigma(x)}\partial_k G_{ij} + 2\sigma_k e^{2\sigma}G_{ij},$$

$$\dot{\partial}_k \overline{G}_{ij} = e^{2\sigma(x)}\dot{\partial}_k G_{ij}, \tag{3.2}$$

and

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \sigma_k + \delta^i_k \sigma_j - G_{jk}G^{im}\sigma_m, \tag{3.3}$$

where $\sigma(k) = \partial_k \sigma(x)$.

The conformal change of the generalized Chern's non-linear connection $N^i_j$ defined by (2.4) is given as follows [10]:

$$\overline{N}^i_j = N^i_j + y^i \sigma_j - f^i_h y^h f^r_j \sigma_r. \tag{3.4}$$

Furthermore, the $hv$-torsion $P^i_{jk}$ defined by $P^i_{jk} = \dot{\partial}_k N^i_j - \Gamma^i_{kj}$ is changed to

$$\begin{aligned}\overline{P}^i_{jk} &= \dot{\partial}_k \overline{N}^i_j - \overline{\Gamma}^i_{kj} \\ &= \dot{\partial}_k (N^i_j + y^i \sigma_j - f^i_h y^h f^r_j \sigma_r) \\ &\quad - (\Gamma^i_{jk} + \delta^i_j \sigma_k + \sigma^i_k \sigma_j - G_{jk} G^{ir} \sigma_r).\end{aligned}$$

Now if we put

$$P_k = P^m_{nk} = \dot{\partial}_k N^m_m - \Gamma^m_{mk}, \tag{3.5}$$

then we get

$$\begin{aligned}\overline{P}_k &= \dot{\partial}_k (N^m_m - y^m \sigma_m - f^m_h y^h f^r_m \sigma_r) \\ &\quad - (\Gamma^m_{mk} + 2n\sigma_k + \delta^m_k \sigma_m - G_{mk} G^{mr} \sigma_r) \\ &= (\dot{\partial}_k N^m_m - \Gamma^m_{mk}) - \delta^m_k \sigma_m - f^m_h \delta^h_k f^r_m \sigma_r \\ &\quad -2n\sigma_k - \sigma_k + \sigma_k \\ &= P_k - \sigma_k + \sigma_k - 2n\sigma_k - \sigma_k + \sigma_k \\ &= P_k - 2n\sigma_k.\end{aligned} \tag{3.6}$$

We suppose that $n \geq 1$. Then we obtain

$$\sigma_k = (P_k - \overline{P}_k)/2n. \tag{3.7}$$

Substituting (3.7) in (3.4), we have

$$\overline{N}^i_j = N^i_j + y^i (P_j - \overline{P}_j)/2n - f^i_h y^h f^r_j (P_r - \overline{P}_r)/2n.$$

The above can be rewritten as

$$\overline{N}^i_j + (y^i \overline{P}_j - f^i_h y^h f^r_j \overline{P}_r)/2n = N^i_j + (y^i P_j - f^i_h y^h f^r_j P_r)/2n.$$

Putting

$$L^i_j = N^i_j + \frac{1}{2n}(y^i P_j - f^i_h y^h f^r_j P_r), \tag{3.8}$$

we get a conformally invariant non-linear connection $L^i_j$, that is, $\overline{L}^i_j = L^i_j$.

Next, substituting (3.7) in (3.3), we have

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j(P_k - \overline{P}_k)/2n + \delta^i_k(P_j - \overline{P}_j)/2n - \overline{G}_{jk}\overline{G}^{im}(P_m - \overline{P}_m)/2n.$$

The above can be written in the form

$$\overline{\Gamma}^i_{jk} + \frac{1}{2n}(\delta^i_j\overline{P}_k + \delta^i_k\overline{P}_j - \overline{G}_{jk}\overline{G}^{im}\overline{P}_m)$$

$$= \Gamma^i_{jk} + \frac{1}{2n}(\delta^i_jP_k + \delta^i_kP_j - G_{jk}G^{im}P_m).$$

Putting

$$L^i_{jk} = \Gamma^i_{jk} + \frac{1}{2n}(\delta^i_jP_k + \delta^i_kP_j - G_{jk}G^{im}P_m), \tag{3.9}$$

we obtain a conformally invariant $h$-connection.

Thus we have

**Theorem 3.1.** *In a Rizza manifold $M^{2n}$, we have a conformal invariant h-Finsler connection $(L^i_{jk}, L^i_j)$ given by* (3.9) *and* (3.8).

### 4. Conformal Change of $(G, L)$-structure

Let $M^{2n}$ be a Rizza manifold satisfying a $(G, L)$-structure, that is to say, $M^{2n}$ admits a generalized Finsler metric $G_{ij}(x, y)$ and a non-linear connection $L^i_j(x, y)$ defined by (2.1) and (3.8), respectively. We put

$$X^*_k = \partial_k - L^m_k\dot{\partial}_m,$$
$$F^{*i}_{jk} = \frac{1}{2}G^{im}(X^*_jG_{mk} + X^*_kG_{mj} - X^*_mG_{jk}), \tag{4.1}$$

then $F^*\Gamma = (F^{*i}_{jk}, L^i_j, C^i_{jk})$ is a symmetric Finsler connection and it is said

to be the *Finsler connection* associated with a $(G, L)$-structure. With respect to $F^*\Gamma$ -connection, (2.6), (2.7) and (2.8) are written as follows:

$$R^{*i}_{jk} = X^*_k L^i_j - X^*_j L^i_k, \quad P^{*i}_{jk} = \dot{\partial}_k L^i_j - F^{*i}_{jk},$$

$$R^{*i}_{hjk} = K^{*i}_{hjk} + C^i_{hm} R^{*m}_{jk}, \quad P^{*i}_{hjk} = \dot{\partial}_k F^{*i}_{hj} - Q^{*i}_{hjk},$$

$$K^{*i}_{hjk} = X^*_k F^{*i}_{hk} + F^{*i}_{mk} F^{*m}_{hj} - j/k,$$

$$Q^{*i}_{hjk} = \nabla^*_j C^i_{hk} - C^i_{hm} P^{*m}_{jk}. \tag{4.2}$$

The conformal changes of (4.1) and (4.2) are expressed as follows:

(a) $\overline{F}^{*i}_{jk} = F^{*i}_{jk} + \sigma_j \delta^i_k + \sigma_k \delta^i_j - \sigma^i G_{jk}$,

(b) $\overline{C}^i_{jk} = C^i_{jk}$,

(c) $\overline{R}^{*i}_{jk} = \overline{X}^*_k \overline{L}^i_j - j/k = X^*_k L^i_j - j/k = R^{*i}_{jk}$,

(d) $\overline{P}^{*i}_{jk} = \dot{\partial}_k \overline{L}^i_j - \overline{F}^i_{kj} = P^{*i}_{jk} - \sigma_j \delta^i_k - \sigma_k \delta^i_j + \sigma^i G_{jk}$,

(e) $\overline{K}^{*i}_{hjk} = \overline{X}^*_k \overline{F}^{*i}_{hk} - \overline{X}^*_j \overline{F}^{*i}_{hk} + \overline{F}^{*i}_{mk} \overline{F}^{*m}_{hj} - \overline{F}^{*i}_{mj} \overline{F}^{*m}_{hk}$

$$= K^{*i}_{hjk} + \delta^i_j \sigma_{kh} - \delta^i_k \sigma_{jh} - G_{hj} \sigma^i_k + G_{hk} \sigma^i_j, \tag{4.3}$$

where

$$\sigma^i = G^{im} \sigma_m, \quad \sigma_{hk} = \nabla^*_k \sigma_h - \sigma_k \sigma_h + \sigma_r \sigma^r G_{hk}/2, \quad \sigma^h_k = G^{hm} \sigma_{mk}.$$

Now it is seen from (b) and (d) of (4.3) that

$$\overline{C}^i_{jm} \overline{P}^{*m}_{k0} = C^i_{jm} P^{*m}_{k0} - \sigma_0 C^i_{jk} + C^i_{jm} \sigma^m y_k, \tag{4.4}$$

which yield

$$\overline{C}_m \overline{P}^{*m}_{k0} = C_m P^{*m}_{k0} - \sigma_0 C_k + C_m \sigma^m y_k, \tag{4.5}$$

where $\sigma_0 = \sigma_m y^m$, $C_m = C^r_{rm}$, $C^k = G^{rk} C_r$ and $y_k = G_{rt}\, y^k$.

On the other hand, we get

$$\overline{C}^k = e^{-2\sigma}C_k, \quad \overline{C}^2 = \overline{C}_r\overline{C}^r = e^{-2\sigma}C^2.$$

Hence we have

$$\overline{C}_m\overline{P}^{*m}_{r0}\overline{C}^r = e^{-2\sigma}(P^{*m}_{r0}C^r - \sigma_0 C^2).$$

Assume that $C^2 \neq 0$. So we get

$$\sigma_0 = C_m P^{*m}_{r0}C^r/C^2 - \overline{C}_m\overline{P}^{*m}_{r0}\overline{C}^r/\overline{C}^2.$$

Putting

$$M = C_m P^{*m}_{r0}C^r/C^2, \quad M_k = \dot{\partial}_k M, \quad M^k = G^{km}M_m,$$

we have

$$\sigma_0 = M - \overline{M}, \quad \sigma_k = M_k - \overline{M}_k, \tag{4.6}$$

Substituting (4.6) in (4.3) we get

$$\overline{C}^i_{jm}\overline{P}^{*m}_{r0} = C^i_{jm}P^{*m}_{r0} - (M - \overline{M})C^i_{jk} + C^i_{jm}G^{mr}(M_r - \overline{M}_r)y_k,$$

that is,

$$\overline{C}^i_{jm}\overline{P}^{*m}_{k0} - \overline{M}\,\overline{C}^i_{jk} + \overline{C}^i_{jm}\overline{G}^{mr}\overline{M}_r y_k$$

$$= C^i_{jm}P^{*m}_{k0} - MC^i_{jk} + C^i_{jm}G^{mr}M_r y_k.$$

Hence, the quantity defined by

$$Q^{*i}_{jk} = C^i_{jm}P^{*m}_{k0} - MC^i_{jk} + C^i_{jm}G^{mr}M_r y_k, \tag{4.7}$$

is invariant under the conformal changes of given $F^*\Gamma$-connection in the Rizza manifold, that is, $\overline{Q}^{*i}_{jk} = Q^{*i}_{jk}$.

Next, by means of (a) of (4.3) and (4.6), we have

$$\dot{\partial}_h\overline{F}^{*i}_{jk} + (\dot{\partial}_h\dot{\partial}G^{im})\overline{M}_m\overline{G}_{jk} - \overline{M}^i\dot{\partial}_h\overline{G}_{jk}$$

$$= \dot{\partial}_h F^{*i}_{jk} + (\dot{\partial}_h G^{im})M_m G_{jk} - M^i\dot{\partial}_h G_{jk}.$$

If we put

$$F^{*i}_{hjk} = \dot{\partial}_h F^{*i}_{jk} + (\dot{\partial}_h G^{im}) M_m G_{jk} - M^i \dot{\partial}_h G_{jk}, \tag{4.8}$$

then the tensor field $F^{*i}_{hjk}(x, y)$ is also invariant under the conformal changes of the given $F^*\Gamma$ -connection in the Rizza manifold.

For the tensor field $Q^{*i}_{hjk} = \nabla^*_j C^i_{hk} - C^i_{km} P^{*m}_{jk}$ we get, from (b) and (d) of (4.3),

$$\begin{aligned}\overline{Q}^{*i}_{hjk} &= \overline{\nabla}^*_j \overline{C}^i_{hk} - \overline{C}^i_{hm} \overline{P}^{*m}_{jk}\\ &= \overline{X}^*_j \overline{C}^i_{hk} + \overline{\Gamma}^i_{mj} \overline{C}^m_{hk} - \overline{\Gamma}^m_{hj} \overline{C}^i_{mk} - \overline{\Gamma}^m_{kj} \overline{C}^i_{hm} - \overline{C}^i_{hm} \overline{P}^{*m}_{jk}\\ &= Q^{*i}_{hjk} + \sigma_m \delta^i_j C^m_{hk} - \sigma_h C^i_{jk} - \sigma^i G_{jm} C^m_{hk} + \sigma^m G_{hj} C^i_{mk} - \sigma^m G_{jk} C^i_{hm}.\end{aligned}$$

Using (4.6), we see that the tensor $\Psi^{*i}_{hjk}(x, y)$ defined by

$$\begin{aligned}\Psi^{*i}_{hjk} &= Q^{*i}_{hjk} + M_m \delta^i_j C^m_{hk} - M_h C^i_{jk}\\ &\quad - M^i G_{jm} C^m_{hk} + M^m G_{hj} C^i_{mk} - M^m G_{jk} C^i_{hm},\end{aligned} \tag{4.9}$$

is invariant under the conformal changes of given $F^*\Gamma$ -connection in Rizza manifold. From $\sigma_i = \sigma_i(x)$, (4.6) leads us to $\dot{\partial}_j \overline{M}_k = \dot{\partial}_j M_k$, that is, the tensor $\dot{\partial}_j M_k$ itself is invariant under the conformal changes of the given $F^*\Gamma$ -connection in Rizza manifold.

In addition to the above, we have

$$\begin{aligned}\overline{\nabla}^*_j \overline{M}_k &= \nabla^*_j M_k - \nabla^*_j \sigma_k - \sigma_k M_j - \sigma_j M_k + \sigma^m M_m G_{jk}\\ &\quad + 2\sigma_j \sigma_k - \sigma_m \sigma^m G_{jk},\end{aligned}$$

from which we have

$$\begin{aligned}\nabla^*_j \sigma_k &= \nabla^*_j M_k - \overline{\nabla}^*_j \overline{M}_k - \sigma_k M_j - \sigma_j M_k + \sigma_m M^m G_{jk}\\ &\quad + 2\sigma_j \sigma_k - \sigma_m \sigma^m G_{jk}.\end{aligned} \tag{4.10}$$

Since $\nabla^*_j\sigma_k = \nabla^*_k\sigma_j$, we have $\overline{\nabla}^*_j\overline{M}_k - \overline{\nabla}^*_k\overline{M}_j = \nabla^*_jM_k - \nabla^*_kM_j$. Namely, the tensor field defined by $\nabla^*_jM_k - \nabla^*_kM_j$ is also invariant under the conformal change of the given $F^*\Gamma$-connection in Rizza manifold.

Finally, on account of (4.10) and (4.6), we have

$$\sigma_{kj} = \nabla^*_jM_k - \overline{\nabla}^*_j\overline{M}_k - M_jM_k + \overline{M}_j\overline{M}_k$$

$$+ \frac{1}{2}M_mM^mG_{jk} - \frac{1}{2}\overline{M}_m\overline{M}^m\overline{G}_{jk}.$$

Hence we put

$$M_{kj} = \nabla^*_jM_k - M_jM_k + \frac{1}{2}M_mM^mG_{jk}.$$

Now, we have $\sigma_{kj} = M_{kj} - \overline{M}_{kj}$, from which

$$\overline{K}^{*i}_{hjk} = K^{*i}_{hjk} + \delta^i_j(M_{kh} - \overline{M}_{kh}) - \delta^i_k(M_{jh} - \overline{M}_{jh})$$

$$- G_{hj}G^{im}(M_{mk} - \overline{M}_{mk}) + G_{hk}G^{im}(M_{mj} - \overline{M}_{mj}).$$

Thus the tensor field defined by

$$\Omega^{*i}_{hjk} = K^{*i}_{hjk} + \delta^i_jM_{kh} - \delta^i_kM_{jh}$$

$$- G_{hj}G^{im}M_{mk} + G_{hk}G^{im}M_{mj} \tag{4.11}$$

is also invariant under the conformal change of the given $F^*\Gamma$-connection in Rizza manifold. Consequently, we conclude

**Theorem 4.1.** *In Rizza manifold satisfying $C \neq 0$, let us put $M = C_mP^{*m}_{r0}C^r/C^2$ and $M_k = \partial_kM$, then the tensors $Q^{*i}_{jk}$, $F^{*i}_{hjk}$, $\Psi^{*i}_{hjk}$, $\Omega^{*i}_{hjk}$ which are given respectively by (4.7), (4.8), (4.9), (4.11) and $\partial_jM_k$, $\nabla^*_jM_k - \nabla^*_kM_j$ are all invariant under conformal change given $F^*\Gamma$-connection.*

## 5. Flatness of $(G, L)$-structure

We shall define the notion of flatness on a Rizza manifold admitting a $(G, L)$-structure given by (2.1) and (3.8) similar to [10].

**Definition 5.1.** In a Rizza manifold $M^{2n}$, if for every point $p$ of $M^{2n}$, there exists a coordinate neighborhood $(U, x^i)$ containing $p$ such that $X^*_k G_{ij} = 0$ holds on $U$, then $(G, L)$-structure is said to be *flat*.

**Definition 5.2.** In a Rizza manifold $M^{2n}$, if for every point $p$ of $M^{2n}$, there exists a coordinate neighborhood $(U, x^i)$ containing $p$ such that $\partial_k G_{ij} = 0$ and $L^m_k \dot{\partial}_m G_{ij} = 0$ holds on $U$, then $(G, L)$-structure is said to be *strongly flat*.

With the above definitions, we have

**Theorem 5.1.** *A $(G, L)$-structure in a Rizza manifold is flat if and only if*

$$K^{*i}_{hjk} = 0, \quad Q^*_{ihjk} + Q^*_{hijk} = 0, \tag{5.1}$$

*where* $Q^*_{ihjk} = G_{hr} Q^{*h}_{ijk}$.

**Proof.** If $(G, L)$-structure is flat, then from Definition 5.1, $M^{2n}$ is covered by system of local coordinate neighbors $\{(U, x^i)\}$ such that $X^*_k G_{ij} = 0$ holds good in each $U$. Hence we see that $F^{*i}_{jk} = 0$ holds in each $U$ from (4.1). By virtue of (4.2) we have $K^{*i}_{hjk} = 0$. From $\dot{\partial}_k F^{*i}_{hj} = 0$ and $(4.2)_4$ we obtain $P^*_{hijk} = -Q^*_{hijk}$. Moreover, from

$$R^*_{hijk} = -R^*_{ihjk}, \quad P^*_{hijk} = -P^*_{ihjk}, \tag{5.2}$$

we have $Q^*_{ihjk} + Q^*_{hijk} = 0$.

Conversely we assume that (5.1) holds good. By the second Bianchi identity for $F^*\Gamma$-connection, we have

$$\nabla^*_j C_{khi} - \nabla^*_k C_{jhi} + C_{jhr} P^{*r}_{ki} - C_{khr} P^{*r}_{ji} - P^*_{jhki} + P^*_{hjki} = 0,$$

that is,

$$Q^*_{khji} - Q^*_{jhki} - P^*_{jhki} - P^*_{khji} = 0. \tag{5.3}$$

Applying the so-called *Christoffel process* with respect to $k$, $h$ and $j$ to (5.3) and using (5.2),

$$2P^*_{jhki} = Q^*_{khji} + Q^*_{hjki} + Q^*_{hkji} - Q^*_{jhki} - Q^*_{kjhi} - Q^*_{jkhi}.$$

From $(5.1)_2$, this is reduced to $P^*_{jhki} = -Q^*_{jhki}$. Hence from $(4.2)_3$, we have $\dot{\partial}_k F^{*i}_{hj} = 0$. Thus $F^{*i}_{jk} = F^{*i}_{jk}(x)$ on $M^{2n}$. And $K^{*i}_{hjk} = 0$ tells us that $M$ is covered by a system of local coordinate neighborhoods such that $F^{*i}_{jk} = 0$ holds on each $U$. Hence $F^{*i}_{jk} = 0$ and $\nabla^*_k G_{ij} = 0$ lead us to $X^*_k G_{ij} = 0$ on each $U$. Therefore the given $(G, L)$-structure is flat.

**Theorem 5.2.** *A* $(G, L)$*-structure in a Rizza manifold is strongly flat if and only if*

$$K^{*i}_{hjk} = 0, \quad Q^*_{ihjk} + Q^*_{hijk} = 0, \quad (C_{mjk} + C_{mkj}) P^{*m}_{jr} y^r = 0, \tag{5.4}$$

*where* $C_{hjk} = G_{jm} C^m_{hk}$.

**Proof.** Let the $(G, L)$-structure be strongly flat. By Definition 5.1 and Definition 5.2, $(G, L)$-structure is eventually flat. By virtue of Theorem 5.1, the former two equations of (5.4) is satisfied. By assumption, $M^{2n}$ is covered by system of local coordinate neighborhoods $\{(U, x^i)\}$ such that, in each $U$, $\partial_k G_{ij} = 0$ and $L^m_k \dot{\partial}_m G_{ij} = 0$ hold. On the other hand, we find, from (2.5) that

$$\dot{\partial}_m G_{jk} = C_{mjk} + C_{mkj}. \tag{5.5}$$

From $(4.2)_2$, we have $P^{*i}_{jr} y^r = L^i_j - F^{*i}_{jr} y^r$. Since $F^{*i}_{jk} = 0$ in $U$, we have $P^{*i}_{jr} y^r = L^i_j$. Therefore, from the definition of the strongly flat and (5.5), we get $(C_{mjk} + C_{mkj}) P^{*m}_{jr} y^r = 0$.

Conversely, we suppose that (5.4) holds good. By virtue of Theorem 5.1, we see that $(G, L)$-structure is flat. Hence, with respect to the assigned coordinate neighborhood $U$ of the flatness, $X_k^* G_{ij} = 0$, from which follows $F_{jk}^{*i} = 0$. Thus, from (4.2), $P_{jr}^{*i} y^r = L_j^i$ holds in each $U$. Therefore, from this equation and (5.5), $(5.4)_3$, we have $L_k^m \dot{\partial}_m G_{ij} = 0$ in each $U$. Since $X_k^* G_{ij} = 0$ is shown, $\partial_k G_{ij} = 0$ is also true in each $U$. Hence the given $(G, L)$-structure is strongly flat.

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# A NOTE ON SIMPLE BCI-ALGEBRAS

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Submitted by Young Bae Jun

## Abstract

In this note we obtain a structural theorem and some properties of simple BCI-algebras.

The notions of BCK-algebras and BCI-algebras were introduced by Imai and Iseki [1, 2]. In 1988, Jiang [3] obtained a structural theorem of finite simple BCK-algebras. That is, an $n(\geq 4)$ order BCK-algebra is simple if and only if its proper subalgebras are all simple. The main aim of this note is to generalize this theorem.

By a BCI-algebra we mean a nonempty set $X$ with a binary operation $*$ and a constant 0 satisfying the following axioms: for any $x, y, z \in X$,

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

For a BCI-algebra $X$, the set $B := \{x \in X \mid 0 \leq x\}$ is called the

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*p-radical* of $X$. A nonempty subset $I$ of a BCI (BCK)-algebra $X$ is called an *ideal* of $X$ if $(I_1)$ $0 \in I$, $(I_2)$ $x * y \in I$ and $y \in I$ imply $x \in I$. An ideal $A$ of a BCI-algebra $X$ is called a *closed ideal* if $A$ is also a subalgebra of $X$. A BCI-algebra $X$ is called *simple* if its ideals are exactly $\{0\}$ and $X$ itself. In this paper, $(S)$ denotes the ideal generated by a nonempty subset $S$ of $X$. We write

$$a * x^n = (\cdots((a * \overbrace{x) * x) * \cdots) * x}^{n}.$$

**Theorem 1** (Structural theorem of simple BCI-algebra). *A BCI-algebra X is simple if and only if its subalgebras are all simple.*

**Proof.** Because $X$ is a subalgebra of itself, so the sufficiency holds. To prove the necessity, we assume that $H$ is a subalgebra of $X$ but not simple. By the definition of simple BCI-algebra, there is a non-trivial ideal $A$ $(A \neq \{0\}, H)$ in $H$. We consider ideal $(A)$ generated in $X$. Since $X$ is simple, we have $(A) = X$. For any $b \in H \setminus A$, then $b \in X = (A)$, by the structure of $(A)$, there are elements $a_1, a_2, \ldots, a_m$ in $A$ such that $((\cdots((b * a_1) * a_2) * \cdots) * a_{m-1}) * a_m = 0 \in A$. Since $A$ is an ideal of $H$, we have $b \in A$. This is a contradiction. Hence $H$ is simple. We have completed the proof.

A natural question arises: Can the condition of Theorem 1 be improved further so that all proper subalgebras of $X$ become simple? That is, can we prove: If all proper subalgebras of $X$ are simple, then $X$ is simple. The following example shows that the answer to this question in general is negative.

**Example.** Let $X = \{0, 1, 2\}$ and operation $*$ be given by the table:

| * | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 |
| 2 | 2 | 2 | 0 |

Then $X$ is a BCI-algebra but not simple, because $A = \{0, 1\}$ is a

non-trivial ideal of $X$. All proper subalgebras of $X$ are simple, because they consist only of one or two elements.

Open problem: Let $X$ be a BCI-algebra. If $|X| \geq 4$ and all proper subalgebras of $X$ are simple, is $X$ simple?

**Theorem 2.** *Let $X$ be a BCI-algebra and $A$ be a maximal closed ideal of $X$. Then $X/A$ is simple.*

**Proof.** If $A$ is a maximal closed ideal of $X$, assume that $T$ is an ideal of $X/A$ and $Y = \{x \mid A_x \in T\}$. Since $A_0 \in T$, we have $0 \in Y$, and so $Y \neq \emptyset$. Let $x, y * x \in Y$, then $A_x, A_{y*x} = A_y * A_x \in T$. As $T$ is an ideal, we have $A_y \in T$, so $y \in Y$. Hence $Y$ is an ideal of $X$. For any $a \in A$, as $A$ is a closed ideal, we have $a * 0 = a \in A$, $0 * a \in A$. Thus $A_a = A_0 \in T$, and so $a \in Y$, which means $A \subseteq Y$. But $A$ is maximal, then $Y = X$ or $Y = A$. If $Y = X$, then $T = X/A$. If $Y = A$, then $T = \{A_0\}$. Hence $X/A$ is a simple BCI-algebra. The proof is complete.

Recall that a BCI-algebra is called a *well BCI-algebra* if its ideals are all closed.

**Theorem 3.** *Let $X$ be a well BCI-algebra and $A$ be an ideal of $X$. Then $X/A$ is simple if and only if $A$ is a maximal ideal of $X$.*

**Proof.** By Theorem 2, the sufficiency holds. To prove the necessity, assume that $B$ is an ideal of $X$ with $A \subseteq B \subseteq X$. Then $B/A$ is an ideal of $X/A$. Since $X/A$ is simple, we have $B/A = X/A$ or $B/A = \{A_0\}$. If $B/A = X/A$, then for any $x \in X$, there exists $y \in B$ such that $A_x = A_y$, which implies $x * y \in A \subseteq B$, and so $x \in B$, thus $B = X$. If $B/A = \{A_0\}$, then for any $y \in B$, $A_y = A_0$, which implies $y = y * 0 \in A$, that is $B = A$. Hence $A$ is a maximal ideal of $X$. The proof is complete.

**Theorem 4.** *Let $X$ be a BCI-algebra and $B$ be its p-radical. If $B$ is a maximal ideal of $X$, then $X$ is a well BCI-algebra.*

**Proof.** Since $B$ is a closed ideal of $X$, by Theorem 2, $X/B$ is simple.

Suppose $A$ is an ideal of $X$, to prove $A$ is closed, it suffices to show that $0 * x \in A$ for every $x \in A$. We consider ideal $(B_x)$ generated in $X/B$. As $X/B$ is simple, $(B_x) = \{B_0\}$ or $(B_x) = X/B$. If $(B_x) = \{B_0\}$, then $B_x = B_0$, and so $x \in B$, $0 * x = 0 \in A$. If $(B_x) = X/B$, then $B_0 * B_x \in (B_x)$. By the structure of $(B_x)$, there exists a nonnegative integer $n$ such that $(B_0 * B_x) * (B_x)^{n-1} = B_0$, that is $B_0 * (B_x)^n = B_0$, $B_{0*x^n} = B_0$. It follows that $0 * x^n \in B$ so that $0 * (0 * x^n) = 0$. Thus $0 * x^n = (0 * (0 * x^n)) * x^n = (0 * x^n) * (0 * x^n) = 0 \in A$. Since $A$ is an ideal, we have $0 * x \in A$. Hence $X$ is a well BCI-algebra and the proof is complete.

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# SOME MODIFICATIONS OF PARACOMPACTNESS

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## Abstract

In this paper, we shall introduce $P_1$-metacompactness as a strong form of metacompactness and $P_2$-metacompactness as a generalization of $P_1$-metacompactness. We shall also use ω-open sets to introduce ω*-paracompactness as a strong form of paracompactness under certain separation axioms. We study each of which and provide some examples related to our results. We also propose some open questions.

## 1. Introduction

Mashhour et al. [4] introduced and studied $P_1$-paracompact and $P_2$-paracompact spaces as strong forms of paracompact spaces. In this paper we shall introduce $P_1$-metacompactness as a strong form of metacompactness and as a generalization of $P_1$-paracompactness. We also introduce the concept of $P_2$-metacompactness as a generalization of $P_1$-metacompactness. We shall also use ω-open sets to introduce a strong form of paracompactness under certain separation axioms.

A space $(X, \tau)$ is paracompact if every open cover has a locally finite open refinement. We follow the notions of [7] except for paracompactness.

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Now, we list some main definitions and results which will be helpful in obtaining the main results.

**Definition 1.1.** A subset $S$ of $(X, \tau)$ is called

(i) An $\alpha$-*set* if $S \subseteq \operatorname{int}(\overline{\operatorname{int}(S)})$.

(ii) A *semi-open set* if $S \subseteq \overline{\operatorname{int}(S)}$.

(iii) A *preopen set* if $S \subseteq \operatorname{int}(\overline{S})$.

We denote the families of all $\alpha$-sets (resp. semi-open sets, preopen sets) by $\tau^\alpha$ (resp. $SO(X, \tau)$, $PO(X, \tau)$).

**Proposition 1.1** [2]. *Let $(X, \tau)$ be a space. Then $\tau^\alpha$ is a topology on $X$ finer than $\tau$, and $\tau^\alpha = \{U - N : U \in \tau$ and $N$ is nowhere dense in $(X, \tau)\}$.*

**Proposition 1.2** [5]. *Let $(X, \tau)$ be a space. Then $\tau^\alpha = SO(X, \tau) \cap PO(X, \tau)$.*

**Proposition 1.3** [2]. *Let $(X, \tau)$ be a space, $A \in \tau^\alpha$ and $S \in PO(X, \tau)$. Then $A \cap S \in PO(X, \tau)$.*

**Definition 1.2** [2]. A space $(X, \tau)$ is said to be *submaximal* if every dense subset is open.

**Proposition 1.4** [2]. *A space $(X, \tau)$ is submaximal if and only if $\tau = PO(X, \tau)$.*

**Definition 1.3** [4]. A space $(X, \tau)$ is called

(i) $P_1$-*paracompact* if every preopen cover for $X$ has a locally finite open refinement.

(ii) $P_2$-*paracompact* if every preopen cover for $X$ has a locally finite preopen refinement.

**Proposition 1.5** [2]. *Let $(X, \tau)$ be a space. Then $(X, \tau)$ is $P_1$-paracompact if and only if $(X, \tau)$ is submaximal and paracompact.*

**Definition 1.4.** A space $(X, \tau)$ is called

(i) $P_1$*-metacompact* if every preopen cover for $X$ has a point finite open refinement.

(ii) $P_2$*-metacompact* if every preopen cover for $X$ has a point finite preopen refinement.

**Definition 1.5** [1]. A point $x$ of a space $X$ is called a *condensation point* of the set $A \subseteq X$ if any arbitrary neighborhood of the point $x$ contains an uncountable subset of this set.

**Definition 1.6** [3]. A subset of a space $X$ is called $\omega$*-closed* if it contains all its condensation points. The complement of an $\omega$-closed set is called $\omega$*-open set*. The collection of all $\omega$-open sets in $X$ will be denoted by $\tau_\omega$. Also, if $A \subseteq X$, then $\underline{A}$ will denote the intersection of all $\omega$-closed sets which contain $A$.

Observe that $A$ is $\omega$-open if and only if for every $x$ in $A$ there is an open set $U$ and a countable subset $C$ such that $x \in U - C \subseteq A$.

One can easily observe that $\omega$-open sets are closed under arbitrary union and finite intersection, i.e., $\tau_\omega$ is indeed a topology on $X$.

## 2. Results

The following characterizations of $P_1$-metacompact and $P_2$-metacompact spaces can be easily established. Their proofs are easy and straightforward.

**Theorem 2.1.** *A space $(X, \tau)$ is $P_1$-metacompact (resp. $P_2$-metacompact) if every preopen cover $\{U_i : i \in I\}$ has a point finite open (resp. preopen) refinement of the form $\{V_i : i \in I\}$ such that $V_i \subseteq U_i$ for each $i \in I$.*

The following results follow from the definitions.

**Theorem 2.2.** *Let $(X, \tau)$ be a space.*

(i) *If $(X, \tau)$ is $P_1$-metacompact, then $(X, \tau)$ is $P_2$-metacompact and metacompact.*

(ii) *If* $(X, \tau)$ *is* $P_i$*-paracompact, then* $(X, \tau)$ *is* $P_i$*-metacompact* ($i =$ 1, 2).

We shall see that the converse of each of the above implications is not true, in general.

**Theorem 2.3.** *Let* $(X, \tau)$ *be a* $P_1$*-metacompact space. Then* $\tau^\alpha = PO(X, \tau)$.

**Proof.** By Proposition 1.2, it suffices to show that $PO(X, \tau) \subseteq SO(X, \tau)$. Let $A \in PO(X, \tau)$ and suppose on the contrary that $A \notin SO(X, \tau)$. If $B = A \cap (X - \overline{\text{int}(A)})$, then $\text{int}(B) = \varnothing$ and $X - B$ is preopen. So $\{B, X - B\}$ is a preopen cover of $X$. By Theorem 2.1, there exists an open cover $\{U_1, U_2\}$ of $X$ such that $U_1 \subseteq B$ and $U_2 \subseteq X - B$. Since $\text{int}(B) = \varnothing$, $U_1 = \varnothing$ and hence $U_2 = X$. Thus $B = \varnothing$, a contradiction.

**Corollary 2.1** [2]. *Let* $(X, \tau)$ *be a* $P_1$*-paracompact space. Then* $\tau^\alpha = PO(X, \tau)$.

**Theorem 2.4.** *Let* $(X, \tau)$ *be a* $P_1$*-metacompact space. Then* $\tau = \tau^\alpha$.

**Proof.** By Proposition 1.1, it suffices to show that every nowhere dense subset of $(X, \tau)$ is closed. Let $N$ be a nowhere dense subset of $(X, \tau)$ and suppose there is a point $x \in \overline{N} \cap (X - N)$. Then $\{X - N, X - \{x\}\}$ is an $\alpha$-open cover, whence a preopen cover of $(X, \tau)$. By assumption, there are open sets $U_1$ and $U_2$ such that $X = U_1 \cup U_2$, $U_1 \subseteq X - N$ and $U_2 \subseteq X - \{x\}$. Since $U_1 \cap N = \varnothing$, $U_1 \cap \overline{N} = \varnothing$ and hence $x \notin (U_1 \cup U_2)$, a contradiction.

**Corollary 2.2** [2]. *Let* $(X, \tau)$ *be a* $P_1$*-paracompact space. Then* $\tau = \tau^\alpha$.

The following nice characterization of $P_2$-metacompact spaces follows from Theorems 2.3 and 2.4 and Proposition 1.4.

**Theorem 2.5.** *Let $(X, \tau)$ be a space. Then $(X, \tau)$ is $P_1$-metacompact if and only if $(X, \tau)$ is submaximal and metacompact.*

Now we list some examples to show that the converse of each implication in Theorem 2.2 is not true, in general.

**Example 2.1.** Let $X$ be any set with cardinality $|X| > 1$ and let $\tau =$ the indiscrete topology on $X$. Since $PO(X, \tau) =$ the discrete topology on $X$, $X$ is $P_2$-metacompact. Since $\{x\}$ is dense and not open in $X$, $X$ is not submaximal, so by Theorem 2.5, $X$ is not $P_1$-metacompact.

If $X$ is infinite, then $X$ is not $P_2$-paracompact. If not, then the preopen cover $\{\{x\} : x \in X\}$ admits a locally finite preopen refinement $\{V_x : x \in X\}$ such that $V_x \subseteq \{x\}$ for each $x \in X$, so $V_x = \{x\}$ for each $x \in X$. Since $\{\{x\} : x \in X\}$ is locally finite, $X$ must be finite, a contradiction. So, $X$ is not $P_2$-metacompact.

**Example 2.2.** There exists a $P_1$-metacompact space which is not $P_1$-paracompact.

**Proof.** Consider the Rational Sequence Topology $(X, \tau)$ (Example 65 of [6]). Let $A \subseteq \mathbf{R}$ such that $\overline{A} = \mathbf{R}$, then the set of rationals $Q \subseteq A$ and hence $(X, \tau)$ is submaximal. It is known that $(X, \tau)$ is metacompact non-paracompact. Therefore, by Proposition 1.5 and Theorem 2.5, $(X, \tau)$ is $P_1$-metacompact non-$P_1$-paracompact.

Finally, we propose the following open questions that are related to our preceding results.

**Question 2.1.** Are $P_2$-metacompact spaces metacompact?

**Question 2.2.** Are $P_2$-metacompact $T_1$ spaces $P_1$-metacompact?

## 3. $\omega^*$-paracompact Spaces

**Definition 3.1.** A space $(X, \tau)$ is $\omega^*$-*paracompact* if every $\omega$-open cover of $X$ has a locally finite $\omega$-open refinement.

**Remark.** If $\{U_i : i \in I\}$ denotes a $\omega$-open cover of $\omega^*$-paracompact space $(X, \tau)$, then there is a locally finite $\omega$-open refinement of the form $\{V_i : i \in I\}$ such that $V_i \subseteq U_i$ for each $i \in I$.

The following results follow easily:

**Theorem 3.1.** (i) *If $(X, \tau)$ is regular $\omega^*$-paracompact, then $(X, \tau)$ is paracompact.*

(ii) *If $(X, \tau)$ is $\omega^*$-paracompact, then $(X, \tau_\omega)$ is paracompact.*

(iii) *If $(X, \tau)$ is Hausdorff and $\omega^*$-paracompact, then $(X, \tau_\omega)$ is normal.*

**Lemma 3.1.** *Let $(X, \tau)$ be a space and let $\{A_\alpha : \alpha \in \Lambda\}$ be a locally finite family of subsets of X. Then $\{\underline{A_\alpha} : \alpha \in \Lambda\}$ is locally finite.*

**Proof.** Let $x \in X$. Choose an open set $G_x$ such that $x \in G_x$ where $G_x$ meets at most finitely many $A_\alpha$, say $A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}$. If $y \in G_x \cap \underline{A_\alpha}$, then $y \in G_x \cap \overline{A}$ and hence $y \in G_x \cap A$. Therefore, for some $k$, $\alpha = \alpha_k$.

**Lemma 3.2.** *If $\{A_\alpha : \alpha \in \Lambda\}$ is a locally finite family of subsets of X, then $\bigcup\{\underline{A_\alpha} : \alpha \in \Lambda\} = \underline{\bigcup\{A_\alpha : \alpha \in \Lambda\}}$.*

**Proof.** Follows since locally finite families in $(X, \tau)$ are locally finite in $(X, \tau_\omega)$.

**Theorem 3.2.** *Let $(X, \tau)$ be a space such that $(X, \tau_\omega)$ is regular. Then the following are equivalent:*

(i) *X is $\omega^*$-paracompact.*

(ii) *Every $\omega$-open cover of $(X, \tau)$ has a locally finite refinement.*

(iii) *Every $\omega$-open cover of X has an $\omega$-closed locally finite refinement.*

**Proof.** (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) Let $\underset{\sim}{U}$ be an ω-open cover of $X$. For each $x \in X$ choose $U_x \in \underset{\sim}{U}$ such that $x \in U_x$ and an ω-open set $V_x$ such that $x \in V_x \subseteq \underline{V_x} \subseteq U_x$. Let $\underset{\sim}{V} = \{V_x : x \in X\}$, then $\underset{\sim}{V}$ has a locally finite refinement $\underset{\sim}{A}$. It follows that $\underset{\sim}{A}^* = \{\underline{A} : A \in \underset{\sim}{A}\}$ is ω-closed cover for $X$ and by Lemma 3.1, $\underset{\sim}{A}^*$ is locally finite. Let $\underline{A} \in \underset{\sim}{A}^*$, then there exists $V_x \in \underset{\sim}{V}$ such that $A \subseteq V_x$, so $\underline{A} \subseteq \underline{V_x} \subseteq U_x \in \underset{\sim}{U}$. Therefore, $\underset{\sim}{A}^*$ refines $\underset{\sim}{U}$.

(iii) $\Rightarrow$ (i) Let $\underset{\sim}{U}$ be an ω-open cover of $X$. By (iii), $\underset{\sim}{U}$ has a locally finite refinement $\underset{\sim}{A} = \{A_\alpha : \alpha \in \Lambda\}$. For each $x \in X$ choose an open neighborhood $V_x$ of $x$ such that $V_x$ meets only finitely many members of $\underset{\sim}{A}$. Now $\underset{\sim}{V} = \{V_x : x \in X\}$ is an open (hence ω-open) cover of $X$. So again, by (iii), it has an ω-closed locally finite refinement $\underset{\sim}{B}$. For each $\alpha \in \Lambda$ let $W_\alpha = X - \bigcup\{B \in \underset{\sim}{B} : B \cap A_\alpha = \varnothing\}$. Then by Lemma 3.2, $W_\alpha$ is ω-open for each α. Let $\underset{\sim}{W} = \{W_\alpha : \alpha \in \Lambda\}$. Then $\underset{\sim}{W}$ is a locally finite ω-open refinement.

**Definition 3.2.** A space $(X, \tau)$ is *locally finite* (*countable*) if every point has a finite (countable) neighborhood.

**Theorem 3.3.** *If $(X, \tau)$ is locally finite space, then $(X, \tau)$ is both ω*-paracompact and paracompact.*

**Proof.** Straightforward.

**Theorem 3.4.** *Let $(X, \tau)$ be a locally countable space. Then $(X, \tau)$ is ω*-paracompact if and only if $(X, \tau)$ is a locally finite space.*

**Proof.** $\Rightarrow$) Suppose $(X, \tau)$ is ω*-paracompact. Consider $\underset{\sim}{U} = \{\{x\} : x \in X\}$ then $\underset{\sim}{U}$ is ω-open cover for $(X, \tau)$. Let $\underset{\sim}{V} = \{V_x : x \in X\}$ be a locally finite open refinement of $\underset{\sim}{U}$ such that $V_x \subseteq \{x\}$. Thus $V_x = \{x\}$

for each $x \in X$ and hence $\underset{\sim}{V} = \underset{\sim}{U}$. For each $x \in X$, choose an open neighborhood $U_x$ which meets finitely many members of $\{\{x\} : x \in X\}$, so $U_x$ must be finite and $(X, \tau)$ is a locally finite space.

$\Leftarrow$) Follows from Theorem 3.3.

**Theorem 3.5.** *$\omega^*$-paracompactness is hereditary with respect to $\omega$-closed sets.*

**Proof.** Let $(X, \tau)$ be a space. Let $A$ be an $\omega$-closed subset of $X$ and let $\underset{\sim}{U}$ be an $\omega$-open cover of $A$. Each $U \in \underset{\sim}{U}$ is open in $(A, (\tau/A)_\omega)$, so for each $U \in \underset{\sim}{U}$ we can find $\omega$-open set $V_u$ in $(X, \tau)$ such that $U = V_u \cap A$. Then $\underset{\sim}{W} = \{V_u : U \in \underset{\sim}{U}\} \cup \{X - A\}$ is an $\omega$-open cover of $X$. Since $(X, \tau)$ is $\omega^*$-paracompact, $\underset{\sim}{W}$ has a locally finite $\omega$-open refinement $\underset{\sim}{B}$. Let $\underset{\sim}{B}^* = \{B \cap A : B \in \underset{\sim}{B} \text{ and } B \cap A \neq \varnothing\}$. Then $\underset{\sim}{B}^*$ is a locally finite $\omega$-open refinement of $\underset{\sim}{U}$ in $A$.

**Corollary 3.1.** *$\omega^*$-paracompactness is hereditary with respect to closed sets.*

Now, we list a sum theorem concerning $\omega^*$-paracompact Hausdorff spaces.

**Theorem 3.6.** *Let $\{F_\alpha : \alpha \in \Lambda\}$ be a locally finite $\omega$-closed covering of a space $(X, \tau)$ such that $F_\alpha$ is $\omega^*$-paracompact for all $\alpha$. Then $(X, \tau)$ is $\omega^*$-paracompact for any Hausdorff space $(X, \tau)$.*

**Proof.** Let $\underset{\sim}{U} = \{U_\delta : \delta \in \Delta\}$ be any $\omega$-open cover for $(X, \tau)$. Then for each $\alpha$, $\{U_\delta \cap F_\alpha : \delta \in \Delta\}$ is an $\omega$-open covering of $F_\alpha$ which is $\omega^*$-paracompact. Therefore, there exists a locally finite (in $F_\alpha$ and hence also in $X$) $\omega$-closed (in $F_\alpha$ and hence also in $X$) refinement $\{B_\beta : \beta \in I^\alpha\}$ of $\{U_\delta \cap F_\alpha : \delta \in \Delta\}$. Let $\underset{\sim}{U} = \{B_\beta : \beta \in I^\alpha, \alpha \in \Lambda\}$. Then $\underset{\sim}{U}$ is locally

finite $\omega$-closed refinement of $\underset{\sim}{U}$. Therefore, by Theorem 3.2, $(X, \tau)$ is $\omega^*$-paracompact.

The following result will be needed in obtaining a compact metric space which is not $\omega^*$-paracompact.

**Proposition 3.1.** *If $(X, \tau)$ is a separable space with the property that every nonempty open subset of $X$ is uncountable, then $(X, \tau_\omega)$ is not regular.*

**Proof.** Suppose on the contrary that $(X, \tau_\omega)$ is regular. Let $D$ be a countable dense subset of $X$ and let $P \notin D$. Since $(X, \tau_\omega)$ is regular and $D$ is closed in $(X, \tau)$, there are $\omega$-open set $U_1$, open set $V$ and a countable set $C_1$ such that $D \subseteq U_1$, $P \in V - C_1$ and $U_1 \cap (V - C_1) = \varnothing$. Since $D$ is dense, $V \cap D \neq \varnothing$. Let $d \in V \cap D$. Since $d \in U_1$, there are an open set $U_2$ and a countable set $C_2$ such that $d \in U_2 - C_2 \subseteq U_1$. Now $U_2 \cap V$ is uncountable and $U_2 \cap V - C_1 \cap D \subseteq U_1 \cap (V - C_1) = \varnothing$. Therefore, $U_2 \cap V \subseteq C_1 \cap D$, a contradiction.

**Example 3.1.** Let $X = [0, 1]$, $\tau =$ the usual topology. Then by the previous result $(X, \tau)$ is not $\omega^*$-paracompact. On the other hand, it is known that $(X, \tau)$ is compact and metric.

**Example 3.2.** There exists a non-$\omega^*$-paracompact space $(X, \tau)$ such that $(X, \tau_\omega)$ is paracompact.

**Proof.** Consider the Irrational Slope Topology $(X, \tau)$ (Example 75 of [6]). Since $\tau_\omega =$ The discrete topology, $(X, \tau)$ is paracompact. On the other hand, $(X, \tau)$ is not $\omega^*$-paracompact by Theorem 3.4.

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# MULTIRESOLUTION ANALYSIS BY THE SOLUTION OF SECOND-KIND INTEGRAL EQUATIONS

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## Abstract

In this paper, we show that if $\phi(x)$ is the solution of the equation $\phi(x) = \lambda \int_{\mathrm{R}} h(2x - y)\phi(y)\,dy$ with $supp\,\hat{h}(\omega) = [-\pi, \pi]$, then $V_j = span\{\phi(2^j x - k) \mid k \in \mathbb{Z}\}$ construct multiresolution analysis in $L^2(\mathrm{R})$.

## 1. Introduction

We observe that drawing attention to construct multiresolution analysis contributed decisively to construct the wavelet decomposition and reconstruction in $L^2(\mathbb{R})$ with new and more efficient approach to make a distinction from the already established [2, 5, 11] ones can be devoted to carrying out further improvements.

In this paper, we will provide the existence and solution devices of

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wavelet functions in $L^2(\mathbb{R})$ by means of eigenvalue problem, for $\lambda = 2$, of the archetypical integral equations, $\phi(x) = \lambda \int_{\mathbb{R}} h(2x - y)\phi(y)dy$, some advantages of which are that in contrast to the previous methods [2, 3, 5], the requirements are not so restrictive as the ones of using the traditional algorithmic approach in solving wavelets in $L^2(\mathbb{R})$ and allow us to describe uniformly the various earlier solution processes of wavelet analysis.

## 2. Preliminaries

Throughout the paper, $L^2(\mathbb{R})$ will denote the Hilbert space of all Lebesgue square integrable functions on $\mathbb{R}$ with inner product

$$\langle f, g\rangle = \int f(x)\overline{g(x)}\,dx,$$

and the norm

$$\| f \| = \left\{ \int_{\mathbb{R}} | f(x) |^2 dx \right\}^{\frac{1}{2}}, \quad f, g \in L^2(\mathbb{R}).$$

The signs $\wedge$ and $\vee$ denote Fourier transform and the inversion, respectively. Operators mean bounded and linear.

Let us recall that a multiresolution analysis is a sequence $(V_j)_{j\in Z}$ of norm-closed subspaces of $L^2(\mathbb{R})$ such that

(i) $V_j \subset V_{j+1}$

(ii) $u(x) \in V_j$ if and only if $u(2x) \in V_{j+1}$

(iii) $u(x) \in V_0$ if and only if $u(x - k) \in V_0$

(iv) $\overline{\bigcup_{j\in Z} V_j} = L^2(\mathbb{R})$

(v) $\bigcap_{j\in Z} V_j = \{0\}$

(vi) There exists a function $\phi \in V_0$, called a *scaling function*, such that the system $\{\phi(x-k)\}_{k\in Z}$ is a Riesz basis of $V_0$. That is, for all $u(x) \in V_0$, $u(x)$ has unique representation as follows: there exist $C_k$ such that $u(x) = \sum_{k\in Z} C_k \phi(x-k)$.

Moreover, there exist constants $A$ and $B$ such that

$$A\| u \|_{L^2}^2 \le \sum_{k\in Z} | C_k |^2 \le B\| u \|_{L^2}^2. \tag{1}$$

As a result, a sequence $\{h_k\}$ exists such that the scaling function satisfies

$$\phi(x) = \sqrt{2} \sum_{k\in Z} h_k \phi(2x-k). \tag{2}$$

By (2), the Fourier transform of the scaling function must satisfy

$$\hat{\phi}(\omega) = H\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right), \tag{3}$$

where $H(\omega) = \frac{1}{\sqrt{2}} \sum_{k\in Z} h_k e^{-i\omega k}$.

Since $\hat{\phi}(0) = \frac{1}{\sqrt{2\pi}}$ [6], we can apply (3) recursively. This yields, at least formally, the product formula

$$\hat{\phi}(\omega) = \prod_{k=1}^{\infty} \hat{h}\left(\frac{\omega}{2^k}\right). \tag{4}$$

## 3. The Existence of Wavelets by Integral Equations

We now provide the existence and solution devices of wavelets in $L^2(R)$ by means of eigenvalue problem, for $\lambda = 2$, of second-kind integral equation:

$$\phi(x) = \lambda \int_R h(2x-y)\,\phi(y)\,dy. \tag{5}$$

In the following, we will construct the solution of the equation (5) in $L^2(\mathbb{R})$ by giving an appropriate restriction for $h(\cdot)$.

First, we begin by giving the following theorem, clarifying the existence of the solution of the above equation (5) for the case of $\lambda = 2$, by taking suitable function $h(x)$ in $L^2(\mathbb{R})$:

**Theorem 3.1.** *In case* $\lambda = 2$, *there exists a solution of* $\phi(x) = \lambda \int_{\mathbb{R}} h(2x - y)\phi(y)\,dy$ *for some* $h(\cdot) \in L^2(\mathbb{R})$.

**Proof.** Taking the Fourier transform of the given equation:

$$\phi(x) = \lambda \int_{\mathbb{R}} h(2x - y)\phi(y)\,dy$$

$$\hat{\phi}(\omega) = \lambda \int_{\mathbb{R}} \left\{h\left[2\left(x - \frac{y}{2}\right)\right]\right\}^{\wedge} \phi(y)\,dy$$

$$= \frac{\lambda}{2}\hat{h}\left(\frac{\omega}{2}\right)\int_{\mathbb{R}} e^{-i\omega\frac{y}{2}}\phi(y)\,dy$$

$$= \frac{\lambda}{2}\hat{h}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right) \tag{6}$$

and

$$\hat{h}(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} h(x)e^{-i\omega x}\,dx. \tag{7}$$

We take the following

$$\hat{h}(\omega) = \begin{cases} \frac{\lambda}{2}H(\omega)\ (= H(\omega),\ \lambda = 2), & \omega \in [-\pi, \pi] \\ 0, & \omega \notin [-\pi, \pi]. \end{cases} \tag{8}$$

From the fact that the solution of the identity (2) in Preliminaries exists, the identities (2) and (5) are equivalent, which completes the proof.

The following theorem clarifies that the solution $\phi(x)$ belongs to $L^2(\mathbb{R})$.

**Theorem 3.2.** *If the solution of* $\phi(x) = \lambda \int_{\mathbb{R}} h(2x - y)\phi(y)\,dy$ *exists, for* $\lambda = 2$, *and if* $\prod_{k=1}^{\infty} \hat{h}\left(\frac{\omega}{2^k}\right)$ *converges in* $L^2(\mathbb{R})$, *then we have* $\phi(x) \in L^2(\mathbb{R}^2)$.

**Proof.** By substituting $\lambda = 2$ in (8) and replacing $\hat{h}(\omega)$ instead of $H(\omega)$ of the identity (3), we obtain the following:

$$\hat{\phi}(\omega) = \hat{h}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right) = \prod_{k=1}^{\infty} \hat{h}\left(\frac{\omega}{2^k}\right)\hat{\phi}(0).$$

On the other hand, since by assumption $\hat{\phi}(\omega)$ converges in $L^2(\mathbb{R})$, we have the desired assertion.

Now, we are ready to describe that the solution of the equation $\phi(x) = \lambda \int_{\mathbb{R}} h(2x - y)\phi(y)\,dy$ constructs the wavelet analysis.

**Theorem 3.3.** *Let* $\phi(x)$ *be the solution of* (5), $supp\,\hat{h}(\omega) = [-\pi, \pi]$, *and* $\hat{\phi}(\omega) \neq 0$, $w \in [-\pi, \pi]$. *Then* $V_j = span\{\phi(2^j x - k) \mid k \in \mathbb{Z}\}$ *constitute a multiresolution analysis.*

**Proof.** (i) From the given equation (5), we have

$$\phi(x) = \lambda \int_{\mathbb{R}} h(2x - y)\phi(y)\,dy. \tag{9}$$

We take (9) into Fourier transform and obtain

$$\hat{\phi}(\omega) = \lambda \int_{\mathbb{R}} \{h(2x - y)\}^{\wedge} \phi(y)\,dy$$

$$= \lambda \int_{\mathbb{R}} \left\{h\left[2\left(x - \frac{y}{2}\right)\right]\right\}^{\wedge} \phi(y)\,dy$$

$$= \lambda \int_{\mathbb{R}} \frac{1}{2}\hat{h}\left(\frac{\omega}{2}\right) e^{-i\omega\frac{y}{2}} \phi(y)\,dy$$

$$= \frac{\lambda}{2}\hat{h}\left(\frac{\omega}{2}\right)\int_{\mathbb{R}} e^{-i\omega\frac{y}{2}}\phi(y)\,dy$$

$$= \frac{\lambda}{2}\hat{h}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right). \tag{10}$$

If $\hat{h}(\omega) = \sum_{k\in\mathbb{Z}} h_k e^{-ik\omega}$, $\omega \in [-\pi, \pi]$, $k \in \mathbb{Z}$, substituting in (10), we have

$$\hat{\phi}(\omega) = \frac{\lambda}{4}\sum_{k\in\mathbb{Z}} h_k e^{-ik\frac{\omega}{2}}\,\hat{\phi}\left(\frac{\omega}{2}\right).$$

By carrying out Fourier transform of

$$\phi(x) = \lambda\sum_{k\in\mathbb{Z}} h_k\phi(2x - k), \tag{11}$$

we find that (10) and (11) are equivalent.

Thus, for any $l \in \mathbb{Z}$,

$$\phi(2^j x - l) = \lambda\sum_{k\in\mathbb{Z}} h_k\phi(2^{j+1}x - 2l - k)$$

$$= \lambda\sum_{k\in\mathbb{Z}} h_k\phi(2^{j+1}x - 2l - k) \in V_{j+1}.$$

Hence, we obtain $V_j \subset V_{j+1}$.

(ii) If $u(x) \in V_j$ and

$$u(x) = \sum_{k\in\mathbb{Z}} C_k\phi(2^j x - k),$$

we have

$$u(2x) = \sum_{k\in\mathbb{Z}} C_k\phi(2^{j+1}x - k) = \sum_{k\in\mathbb{Z}} C_k\phi(2^{j+1}x - k).$$

Thus $u(2x) \in V_{j+1}$.

(iii) Suppose that $u(x) \in V_0$. Then

$$u(x) = \sum_{k\in\mathbb{Z}} C_k\phi(x - k)$$

and

$$u(x-l) = \sum\nolimits_{k\in\mathbb{Z}} C_k \phi(x-l-k)$$

$$= \sum\nolimits_{k\in\mathbb{Z}} C_{n-l} \phi(x-n) \qquad (\because l+k=n).$$

Hence, $u(x-l) \in V_0$.

(iv) We prove in two steps: first, from (11),

$$\phi\left(2^j x - \frac{l}{2}\right) = \lambda \sum_{k\in\mathbb{Z}} h_k \phi(2^{j+1}x - l - k). \tag{12}$$

That is, $\phi\left(2^j x - \frac{l}{2}\right)$ is represented by the basis of $\phi(2^{j+1}x - k)$, $k \in \mathbb{Z}$.

Thus

$$\left\{\phi\left(2^j x - \frac{k}{2}\right) \mid j, k \in \mathbb{Z}\right\} \subset \{\phi(2^{j+1}x - k) \mid j, k \in \mathbb{Z}\}.$$

Conversely, since

$$\{\phi(2^{j+1}x - k) \mid j, k \in \mathbb{Z}\} \subset \left\{\phi\left(2^j x - \frac{k}{2}\right) \mid j, k \in \mathbb{Z}\right\},$$

we have the following:

$$\{\phi(2^{j+1}x - k) \mid j, k \in \mathbb{Z}\} = \left\{\phi\left(2^j x - \frac{k}{2}\right) \mid j, k \in \mathbb{Z}\right\}.$$

Next, we show that $\left\{\phi\left(2^j x - \frac{k}{2}\right) \mid j, k \in \mathbb{Z}\right\}$ is complete in $L^2(\mathbb{R})$. For any $u \in L^2(\mathbb{R})$, suppose that

$$\left\langle u(x), \phi\left(2^j x - \frac{k}{2}\right)\right\rangle_{L^2(\mathbb{R})} = 0,$$

that is, it is assumed that

$$\left\langle \hat{u}(\omega), \left\{\phi\left(2^j x - \frac{k}{2}\right)\right\}^{\wedge}\right\rangle_{L^2(\mathbb{R})} = 0.$$

Accordingly, we have

$$\int_{\mathbb{R}} \hat{u}(\omega)\overline{\left\{\phi\left(2^j x - \frac{k}{2}\right)\right\}^{\wedge}} d\omega = \int_{\mathbb{R}} \hat{u}(\omega)\left(\frac{1}{2^j}\right)\overline{\hat{\phi}}\left(\frac{\omega}{2^j}\right) e^{ik\frac{\omega}{2^{j+1}}} d\omega.$$

Now, we put $t = \dfrac{\omega}{2^j}$, and obtain

$$\int_{\mathbb{R}} \hat{u}(2^j t)\,\overline{\hat{\phi}}(t)\, e^{ik\frac{t}{2}} dt = 0.$$

Then, by (6), since $supp\,\hat{\phi}(\omega) = [-2\pi, 2\pi]$, we have

$$\int_{\mathbb{R}} \hat{u}(\omega)\overline{\left\{\phi\left(2^j x - \frac{k}{2}\right)\right\}^{\wedge}} d\omega$$

$$= \int_{[-2\pi,\, 2\pi]} \hat{u}(2^j \omega)\,\overline{\hat{\phi}}(\omega)\, e^{ik\frac{\omega}{2^{j+1}}} d\omega = 0, \quad j, k = \mathbb{Z}. \tag{13}$$

Thus

$$\hat{u}(2^j \omega)\,\overline{\hat{\phi}}(\omega) = 0, \quad \omega \in [-2\pi, 2\pi]. \tag{14}$$

Also, by (6) and the continuity of $\hat{\phi}(\omega)$, we have

$$\hat{\phi}(\omega) \neq 0,\ \omega \in [-2\pi, 2\pi].$$

Consequently, $\hat{u}(2^j \omega) = 0,\ \omega \in [-2\pi, 2\pi]$, i.e.,

$$\hat{u}(w) = 0, \quad w \in [-2^{j+1}\pi, 2^{j+1}\pi]. \tag{15}$$

On the other hand, since $j$ is arbitrary, we conclude that $\hat{u}(\omega) = 0$, $w \in \mathbb{R}$, i.e., $u(x) = 0$, $x \in \mathbb{R}$. Therefore, $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$.

(v) We now show that $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$. For all $\lambda(x) \in \bigcap_{j \in \mathbb{Z}} V_j$, we let

$$\lambda(x) = \sum_{k \in \mathbb{Z}} C_{jk}\phi(2^j x - k), \tag{16}$$

Applying the Fourier transform, we have

$$\hat{\lambda}(\omega) = \frac{1}{2^j}\sum_{k\in\mathbb{Z}} C_{jk}\hat{\phi}\left(\frac{\omega}{2^j}\right)e^{-ik\frac{\omega}{2^j}} = \frac{1}{2^j}\hat{\phi}\left(\frac{\omega}{2^j}\right)f\left(\frac{\omega}{2^j}\right), \tag{17}$$

where $f\left(\frac{\omega}{2^j}\right) = \sum_{k\in\mathbb{Z}} C_{jk}e^{-ik\frac{\omega}{2^j}}$.

In this case, if we take $j$ to be sufficiently large negative integer, then $\hat{\phi}\left(\frac{\omega}{2^j}\right) = 0$ is satisfied, so we have $\hat{\lambda}(\omega) = 0$, which implies $\lambda(x) = 0$.

(vi) Finally, we need to prove that $\phi(x-k)$, $k \in \mathbb{R}$ forms a Riesz basis. Let $u(x) = \sum_{k\in\mathbb{Z}} C_k\phi(x-k)$, for all $u(x) \in V_0$. We apply the Fourier transform and obtain

$$\hat{u}(\omega) = H(\omega)\,\hat{\phi}(\omega),$$

where $H(\omega) = \sum_{k\in\mathbb{Z}} C_k e^{-ik\omega}$.

On the other hand, we consider

$$\begin{aligned}\| u \|^2 = \| \hat{u} \|^2 &= \int_{\mathbb{R}} |\hat{u}(\omega)|^2 d\omega \\ &= \int_{\mathbb{R}} |H(\omega)|^2 |\hat{\phi}(\omega)|^2 d\omega \\ &= \sum_{k\in\mathbb{Z}} \int_{2k\pi}^{2(k+1)\pi} |H(\omega)|^2 |\hat{\phi}(\omega)|^2 d\omega \\ &= \sum_{k\in\mathbb{Z}} \int_{[0,\,2\pi]} |H(t+2\pi k)|^2 |\hat{\phi}(t+2\pi k)|^2 dt, \end{aligned} \tag{18}$$

where $\omega - 2\pi k = t$.

Since

$$H(t+2\pi k) = \sum_{k\in\mathbb{Z}} C_k e^{-ik(t+2\pi k)}$$

$$= \sum_{k \in Z} C_k e^{-ikt} e^{-ik2\pi k}$$

$$= \sum_{k \in Z} C_k e^{-ikt} \quad (\because e^{-ik2\pi k} = 1) = H(t),$$

the equation (18) implies that

$$\sum_{k \in Z} \int_{[0,\, 2\pi]} | H(t) |^2 | \hat{\phi}(t + 2\pi k) |^2 dt$$

$$= \int_{[0,\, 2\pi]} | H(t) |^2 \sum_{k \in Z} | \hat{\phi}(t + 2\pi k) |^2 dt. \tag{19}$$

Since $supp\, \hat{\phi}(\omega) = [-2\pi, 2\pi]$, $k$ must be chosen as 0 or 1.

Then

$$\sum_{k \in Z} | \hat{\phi}(t + 2\pi k) |^2 = | \hat{\phi}(t) |^2 + | \hat{\phi}(t + 2\pi k) |^2.$$

Hence the equation (19) becomes

$$\int_{[0,\, 2\pi]} | H(t) |^2 \{ | \hat{\phi}(t) |^2 + | \hat{\phi}(t + 2\pi k) |^2 \} \, dt. \tag{20}$$

We now denote the parenthesis part of the integrand in the equation (20) by $\Omega = | \hat{\phi}(t) |^2 + | \hat{\phi}(t + 2\pi k) |^2$.

And since the Fourier transform $\hat{\phi}(\omega)$ is continuous and $supp\, \hat{\phi}(\omega) = [-2\pi, 2\pi]$, $\Omega$ has clearly the Riesz positive bounds $\frac{1}{A}$ and $\frac{1}{B}$.

Therefore, we can write

$$\frac{1}{A} \geq \sup_{t \in [0,\, 2\pi]} \{\Omega\}, \quad \frac{1}{B} \leq \inf_{t \in [0,\, 2\pi]} \{\Omega\}.$$

Thus we have

$$\frac{1}{A} \int_{[0,\, 2\pi]} | H(t) |^2 dt \geq \int_{[0,\, 2\pi]} | H(t) |^2 \{\Omega\} \, dt$$

$$= \int_{[0,\ 2\pi]} |\hat{u}(w)|^2 dw = \|u\|^2, \tag{21}$$

$$\frac{1}{B}\int_{[0,\ 2\pi]} |H(t)|^2 dt \le \int_{[0,\ 2\pi]} |H(t)|^2 \{\Omega\}\, dt$$

$$= \int_{[0,\ 2\pi]} |\hat{u}(w)|^2 dw = \|u\|^2. \tag{22}$$

By combining (21) and (22), we have

$$B^{-1}\int_{[0,\ 2\pi]} |H(t)|^2 dt \le \int_{\mathbb{R}} |\hat{u}(w)| dw \le A^{-1}\int_{[0,\ 2\pi]} |H(t)|^2 dt. \tag{23}$$

Evaluating the inequalities of equation (23), we have

$$2\pi\sum_{k\in\mathbb{Z}} |C_k|^2 = \int_{[0,\ 2\pi]} |H(t)|^2 \ge A\int_{\mathbb{R}} |\hat{u}(w)|^2 dw = A\|u\|^2$$

$$2\pi\sum_{k\in\mathbb{Z}} |C_k|^2 = \int_{[0,\ 2\pi]} |H(t)|^2 \le B\int_{\mathbb{R}} |\hat{u}(w)|^2 dw = B\|u\|^2.$$

Thus

$$A\|u\|^2_{L^2(\mathbb{R})} \le \sum_{k\in\mathbb{Z}} |C_k|^2 \le B\|u\|^2_{L^2(\mathbb{R})}.$$

Hence, we obtain that $\phi(x-k)$, $k \in \mathbb{Z}$, is a Riesz basis of $V_0$, which completes the proof.

From these descriptions we readily obtain the following theorem constructing multiresolution analysis:

**Theorem 3.4.** *If* $\phi(x)$ *is the solution of the equation* $\phi(x) = \lambda\int_{\mathbb{R}} h(2x-y)\phi(y)\,dy$ *and* $\operatorname{supp}\hat{h}(w) = [-\pi, \pi]$, *then* $V_j = span\{\phi(2^j x - k) \mid k \in \mathbb{Z}\}$ *construct multiresolution analysis.*

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# ON THE EXACT LIMIT CYCLE FOR LIÉNARD-TYPE EQUATION

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## Abstract

In this paper, exact expressions for the periodic solutions in the phase plane for Liénard-type equations are constructed. We study their qualitative behaviour in view of the uniqueness and stability. We illustrate this by examples.

## 1. Introduction

Consider the following generalized Liénard type equation

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0, \quad \dot{x} = \frac{dx}{dt}, \tag{1}$$

where

$$f(x, \dot{x}) = \sum_{i=0}^{n} f_{i+1}(x)\dot{x}^i.$$

Following [1], this equation will be called as a *Liénard-type equation* of order $n$.

When $n = 0$ or $n = 1$, the equation (1) can be written in the following form

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$$\ddot{x} + f_1(x)\dot{x} + g(x) = 0, \tag{2}$$

$$\ddot{x} + f_2(x)\dot{x}^2 + f_1(x)\dot{x} + g(x) = 0. \tag{3}$$

The most interesting solution of (2) and (3) is the limit cycle in the phase plane. The limit cycle is an isolated closed path corresponding to a periodic solution, and of course independent of the initial conditions $x(0)$ and $\dot{x}(0)$. All other solutions approach or recede from the limit cycle either from its interior or from its exterior (depending on the initial conditions) asymptotically as $t \to \infty$; these solutions are of relatively little interest, in comparison with the periodic solution.

The determination of the exact limit cycle (the periodic solution) of equation (2) or (3), is rare, even if analytically possible.

In [4], A. A. Mostafa has found an exact expression for the limit cycle of the equation (2) by choosing appropriate functions $f_1$ and $g$. In [2] and [3] Guidorizzi establishes sufficient conditions for the existence of a periodic solution for the equations (2) and (3) by using a family of positive definite functions. In [6], Zhou Jin establishes sufficient conditions for existence, uniqueness and stability of a periodic solution for equation (3).

This paper is devoted to determine exact expression for the limit cycle in the phase plane for the equation (3) by choosing appropriate functions $f_1$, $f_2$ and $g$. We also establish sufficient conditions of uniqueness and stability for this limit cycle. We shall present two examples which illustrate applications of our results and many more examples could be easily constructed.

## 2. Auxiliary Lemmas

In order to prove our main results on the determination of the exact expression for the limit cycle for equation (3) appropriate functions $f_1$, $f_2$ and $g$, we shall utilize the known results for a Liénard system in [2], [4], [5] and [6], which we state as the following lemmas.

**Lemma 1** (See [4]). *Let p and q be two polynomials. If*

(i) *$p$ is an even polynomial, with $0 < p(0)$, and there exists a constant $\alpha > 0$ such that with $p(\alpha) = 0$ and $p(x) < 0$ for all $x$ satisfying $0 < \alpha < x$,*

(ii) *$q$ is an odd polynomial, then the equation*

$$\ddot{x} - \left(\frac{3}{2} p'q + pq'\right)\dot{x} + \frac{1}{2} p'(pq^2 - 1) = 0, \quad (\ )' = \frac{d(\ )}{dx} \tag{4}$$

*admits at least one periodic solution with exact expression*

$$\dot{x} = p(x)\, q(x) \pm \sqrt{p(x)}, \tag{5}$$

*located between the lines $x = -\alpha$ and $x = \alpha$.*

From now we put

$$F_1(x) = \int_0^x f_1(s)\, ds; \quad G(x) = \int_0^x g(s)\, ds.$$

**Lemma 2** (See [6, Corollaries 1 and 2 for Theorem 2]). *Suppose that*

(i) *$f_1$, $g$ are of class $C^1$,*

(ii) *$xg(x) > 0$ for $x \neq 0$,*

(iii) *there exist $x_1$ and $x_2$, $x_2 \leq 0 \leq x_1$, and a constant $b \geq 0$ such that $F_1(x)/G^b(x)$ is nondecreasing for $x \notin [x_2, x_1]$, and $F_1(x)/G^b(x) \neq$ const., for $0 < |x| << 1$.*

*Then the equation (2) has at most one limit cycle. If such limit cycle exists, then it is also stable.*

## 3. Results

We now state our main results.

**Theorem 3.** *Consider the equation*

$$\ddot{x} + f_2\dot{x}^2 - \left(\frac{3}{2} p'q + pq' + 2f_2pq\right)\dot{x} + \left(\frac{1}{2} p' + f_2p\right)(pq^2 - 1) = 0. \tag{6}$$

*Suppose that*

(i) *$p$ and $q$ satisfy the assumptions of Lemma* 1;

(ii) *$f_2$ is a continuous function of $\mathbb{R}$ in $\mathbb{R}$.*

*Then the equation* (6) *admits at least one nontrivial periodic solution with exact expression* (5) *located between the lines $x = -\alpha$ and $x = \alpha$.*

**Proof.** Instead of invoking an existence theorem, we shall construct an exact solution of (6). The equation (6) is equivalent to

$$\ddot{x} - \left(\frac{3}{2}p'q + pq'\right)\dot{x} + \frac{1}{2}p'(pq^2 - 1) + f_2(\dot{x}^2 - 2pq\dot{x} + p^2q^2 - p) = 0.$$

We observe that a solution is certainly a solution of (6) if it satisfies

$$\ddot{x} - \left(\frac{3}{2}p'q + pq'\right)\dot{x} + \frac{1}{2}p'(pq^2 - 1) = 0; \tag{7}$$

and at the same time

$$f_2(\dot{x}^2 - 2pq\dot{x} + p^2q^2 - p) = 0. \tag{8}$$

From Lemma 1, (5) is a solution of equation (7).

Substituting (5) into (8), it is easy to see that (5) is also a solution of (8).

This completes the proof of Theorem 3.

**Remark 1.** Let us contrast Theorem 1 with Guidorizzi's result (see [2, Theorem 3]). Our conditions are less restrictive than Guidorizzi's conditions in some sense; in particular, we have proved existence without the hypothesis $xg(x) > 0$ and the assumptions on $f_2$ may be relaxed.

Now, we shall give a sufficient conditions for the uniqueness and stability of limit cycle for equation (6) by transforming it to a Liénard equation of type (2).

**Lemma 4.** *Suppose that $f_2$ is a continuous of $\mathbb{R}$ in $\mathbb{R}$. Then the equation* (6) *can be transformed into a Liénard system as follows:*

$$\begin{cases} \dot{x} = y - F_2(x) \\ \dot{y} = -g_1(x) \end{cases}, \tag{9}$$

*where*

$$F_2(x) = -\int_0^x \left(\frac{3}{2} p'q + pq' + 2f_2pq\right) e^{\int_0^s f_2(u)\,du} ds,$$

$$g_1(x) = \left(\frac{1}{2} p' + f_2p\right)(pq^2 - 1) e^{2\int_0^x f_2(s)\,ds}.$$

**Proof.** It is well known that equation (6) is equivalent to the system

$$\begin{cases} \dot{x} = v \\ \dot{v} = \left(\frac{3}{2} p'q + pq' + 2f_2pq\right)v - f_2(x)v^2 - \left(\frac{1}{2} p' + f_2p\right)(pq^2 - 1). \end{cases} \tag{10}$$

For system (10), by applying the following transformation

$$\begin{cases} x = x \\ v = \left(-\int_0^x f_1(s) e^{\int_0^s f_2(u)\,du} ds + y\right) e^{\int_0^x f_2(s)\,ds} \end{cases}$$

we get

$$\begin{cases} \dot{x} = \left(y + \int_0^x \left(\frac{3}{2} p'q + pq' + 2f_2pq\right) e^{\int_0^s f_2(u)\,du} ds\right) e^{\int_0^x f_2(s)\,ds} \\ \dot{y} = -\left(\frac{1}{2} p' + f_2p\right)(pq^2 - 1) e^{\int_0^x f_2(s)\,ds}. \end{cases} \tag{11}$$

Using the time transformation $d\tau = e^{\int_0^x f_2(s)\,ds} dt$ and still denoting $\tau$ by $t$, the system (11) can be transformed into Liénard system (9), which proves the desired result.

From this lemma we can conclude that equation (6) is equivalent to

$$\ddot{x} + f_3(x)\dot{x} + g_1(x) = 0, \tag{12}$$

where

$$f_3(x) = -\left(\frac{3}{2}p'q + pq' + 2f_2pq\right)e^{\int_0^x f_2(s)\,ds},$$

$$g_1(x) = \left(\frac{1}{2}p' + f_2p\right)(pq^2 - 1)e^{2\int_0^x f_2(s)\,ds}.$$

Next we shall prove the following:

**Theorem 5.** *Suppose that*

(i) *$p$ and $q$ satisfy the assumptions of Lemma 1;*

(ii) *$f_2$ is of class $C^1$ such that*

(a) *there exist $x_1$ and $x_2$, $x_2 \le 0 \le x_1$ such that $\left(\frac{3}{2}p'q + pq' + 2f_2pq\right) \le 0$ for $x \notin [x_2, x_1]$*

(b) $x\left(\frac{1}{2}p' + f_2p\right)(pq^2 - 1) > 0$ *for* $x \neq 0$.

*Then equation (6) has a unique stable limit cycle with exact expression (5).*

**Proof.** Obviously, under the assumptions of theorem, (5) is a periodic solution for equation (6).

In addition, we have $f_3$, $g_1$ are of class $C^1$; and $xg_1(x) > 0$ for $x \neq 0$.

Now, if we put

$$F_3(x) = \int_0^x f_3(s)\,ds; \qquad G_1(x) = \int_0^x g_1(s)\,ds;$$

we get

$$F_3'(x) = f_3(x) \ge 0 \quad \text{for} \quad x \notin [x_2, x_1];$$

then $F_3(x)$ is nondecreasing for $x \notin [x_2, x_1]$.

Using Lemma 2 for $b = 0$, we can conclude that equation (12) admits (5) as a unique stable limit cycle.

Finally from Lemma 4, we can conclude that equation (6) admits (5) as a unique stable limit cycle.

This completes the proof of Theorem 5.

## 4. Examples

**Example.** If we choose

$$p(x) = 1 - x^4, \quad q(x) = x, \quad f_2(x) = cx, \quad c \text{ is a real constant,}$$

then

$$f_1(x) = 7x^4 + 2cx^6 - cx^2 - 1,$$

$$g(x) = 2cx^{11} + 2x^9 - 4cx^7 + 2(c-1)x^5 + 2(c+1)x^3 - 2cx.$$

From Theorem 3, the equation

$$\ddot{x} + cx\dot{x}^2 + (7x^4 + 2cx^3 - 2cx - 1)\dot{x} + g(x) = 0 \tag{13}$$

admits

$$\dot{x} = x(1 - x^4) \pm \sqrt{1 - x^4}$$

as a periodic solution, located between the lines $x = -1$, and $x = 1$.

**Example.** If we choose

$$p(x) = a(4 - x^2), \quad q(x) = bx, \quad f_2(x) = cx,$$

where $a$, $b$ and $c$ are positive constants such that

$$\frac{4ab^2}{16ab^2 + 1} < c < \frac{1}{8}.$$

Then

$$f_1(x) = 2abcx^4 + 4ab - 8abcx^2 - 4ab,$$

and

$$g(x) = a^2b^2cx^7 + a^2b^2(1-8c)x^5$$
$$+ (16a^2b^2c - 4a^2b^2 + ac)x^3 + (a - 4ac)x.$$

It is easy to verify that all the conditions of Theorem 5 are satisfied. Hence, the equation

$$\ddot{x} + cx\dot{x}^2 + f_1(x)\dot{x} + g(x) = 0 \tag{14}$$

admits

$$\dot{x} = abx(4 - x^2) \pm \sqrt{b(4 - x^2)}$$

as a unique stable limit cycle, located between the lines $x = -2$, and $x = 2$.

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# USE OF PARALLEL PROCESSING IN OPTIMIZATION

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## Abstract

Current computer-aided control system design environments seldom support optimization methods for controller design in a truly interactive manner. A prototype tool, called PARSIM, supporting parallel processing, optimization, and a graphical user interface is presented, addressing many of the problems inherent in current approaches to multiobjective optimization-based design methods. An XWindows interface is used to simplify problem formulation and control the optimization processes. Using a previously developed interface, it is shown how the computational burden may be alleviated by parallel processing.

## Introduction

Multiobjective Optimization (MO) is an attractive and powerful approach to designing control systems to satisfy many, often competing, design specifications [10]. However, parametric optimization methods are numerically intensive, and, to be truly effective, require the intervention of the designer to guide the optimization by selecting the appropriate trade-offs that must be made. Further, the effort involved in reformulating a system description into a form suitable for the

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application of standard numerical routines may make MO less appealing.

In the class of problems we consider, the structure of the controller will already be specified, and the control engineer will wish to determine a set of design parameters for this structure. We seek to realize a truly interactive computer-aided control system design (CACSD) environment for optimization through the use of parallel processing, to relieve the computational burden on numerical routines, and graphical interfaces to support and guide the engineer in the optimization process. The influential paper by McFarlane et al. [8] supports a number of aspects of this approach: the use of parallel processing allied to the search for pareto-optimal solutions; and new user interface technologies to aid faster interaction between the designer and the primary design tools. Here, we present a prototype tool, PARSIM, that combines parallel processing, optimization, and a graphical user interface within the framework of a commercial CACSD package, MATLAB [9].

## PARSIM

PARSIM is a collection of software tools that integrate a parallel processing gateway, a set of parallel routines for linear system simulation, and a graphical user interface to the Optimization Toolbox [5]. Fig. 1 shows how the components of the system are combined within the framework of the MATLAB package.

The plant model and associated controller structure are entered using the SIMULINK graphical editor, with the free system variables represented by variables in the MATLAB workspace. An interface to the Optimization Toolbox, written in the MATLAB (version 4) command language, is used to select the optimization parameters, specify the desired goals, call the appropriate optimization procedure, and analyze the results. During the optimization, the objective functions may be evaluated in a number of ways on the parallel platform by MATLAB calls to *resident simulation routines* or by calls to MATLAB *m*- or *mex-files*.

### Optimization in CACSD

In CACSD, optimization is an attractive tool in the selection of design

parameters to satisfy specific objective(s). The usual approach is to aggregate the initial design goals into a single cost function, say $f(\underline{x})$, where $\underline{x}$ is a vector of design parameters, and solve the following nonlinear programming (NP) problem:

$$\min_{\underline{x}\in\Omega} f(\underline{x}), \tag{1}$$

where $\Omega$ defines the set of feasible parameters. However, this is an unsatisfactory approach since the distinction between the various objectives lost in the aggregation process and the designer may be unable to exercise effective design control in the formulation of the optimization problem.

Multiobjective optimization permits the separate design objectives to be simultaneously optimized, allowing the designer to visualize the trade-off between the different goals. This can be expressed as:

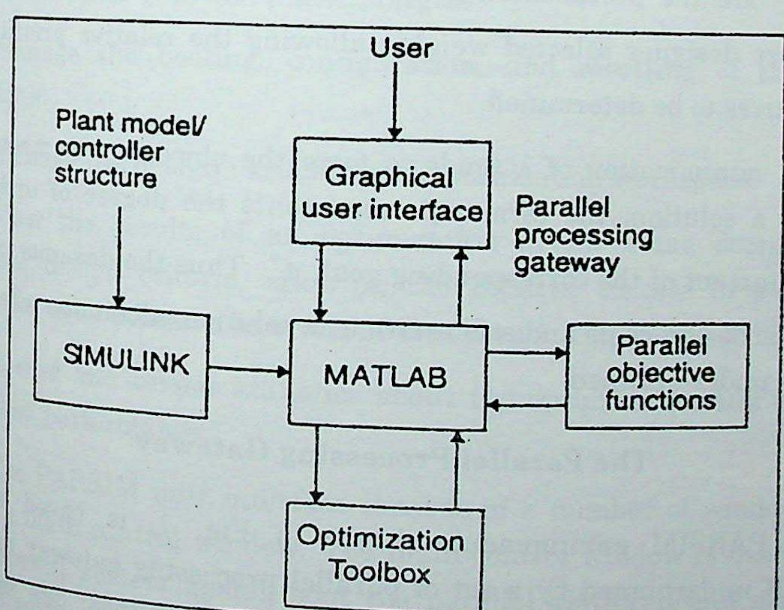


**Fig. 1.** The PARSIM environment.

$$\min_{\underline{x}\in\Omega} f(\underline{x}), \tag{2}$$

where $f = [f_1, f_2, \ldots, f_n]$ is the family of objective functions.

Clearly, there will be no unique solution to the problem posed in (2). Rather, there is a non-inferior solution set for which an improvement in one of the objectives will necessitate the degradation of one or more of the other objectives. The outcome of a MO design process will therefore be a trade-off surface from which the design engineer will be able to select a suitable compromise solution.

Following a survey of alternative MO strategies, the goal attainment method of Gembicki [4] was found to be the most suitable for use in CACSD problems and is available in the Optimization Toolbox. The MO problem, (2), is recast as the following NP problem:

$$\min_{\lambda,\ \underline{x}\in\Omega} \lambda, \tag{3}$$

such that

$$f_i - w_i\lambda \le f_i^*, \qquad i = 1, 2, \ldots, n, \tag{4}$$

where $f_i^*$ are the prespecified goals for the design objectives, $f_i$, and $w_i \ge 0$ are designer selected weights allowing the relative priorities of the objectives to be determined.

Here, minimization of $\lambda$ tends to force the objectives to meet their goals. At a solution, the term $w_i\lambda$ represents the degree of under- or over-attainment of the corresponding goal, $f_i^*$. Thus the designer can set unrealistic design goals and still arrive at a solution, albeit one where the goals are under-attained.

## The Parallel Processing Gateway

The PARSIM environment, shown in Fig. 1, is based around MATLAB underpinned by a set of parallel processing gateway routines [2]. The gateway itself consists of two sets of routines: one set that controls the communications between the host work-station and the compute nodes, and another set that distributes the incoming data and outgoing results to and from the individual compute nodes on the parallel platform. This is made possible by MATLAB's mex-file interface, a facility for linking pre-compiled C and FORTRAN code dynamically into the

MATLAB address space. A number of other routines have been developed for booting, configuration, and loading the platform from within the MATLAB environment. In PARSIM, the gateway is used to configure and boot the parallel platform and call *parallel objective function routines* through the user interface.

## The User Interface

The user interface to PARSIM is designed to allow the user to perform the following tasks:

- Optimize the design of a control system consisting of a plant model and controller structure expressed in a SIMULINK block diagram.
- Express the design criteria of the system in the time domain, e.g., overshoot, rise time, settling time.
- Express the design criteria of the system in the frequency domain, e.g., bandwidth, gain and phase margins.
- Manage the booting, configuration, and resetting of the parallel platform.
- Manage the design variables and optimization workspace.
- View the results of an optimization in the same domain as the original design criteria, allowing the relative success of each design objective to be assessed in terms of the overall design.
- Record and review statistics about the optimization and simulation routines' performance.

The PARSIM user interface consists of a number of windows driven from a main control window. The main control window is used to select the current system model, boot the parallel platform, and manage the user's workspace. The optimization control window, shown in Fig. 2, is used to manage all aspects of the design of a system controller via optimization. The main pull-down menus at the top of the window are used to load and save the design objectives and call the *Objective Editor Window*, Fig. 3, select the optimization routine and invoke the optimization.

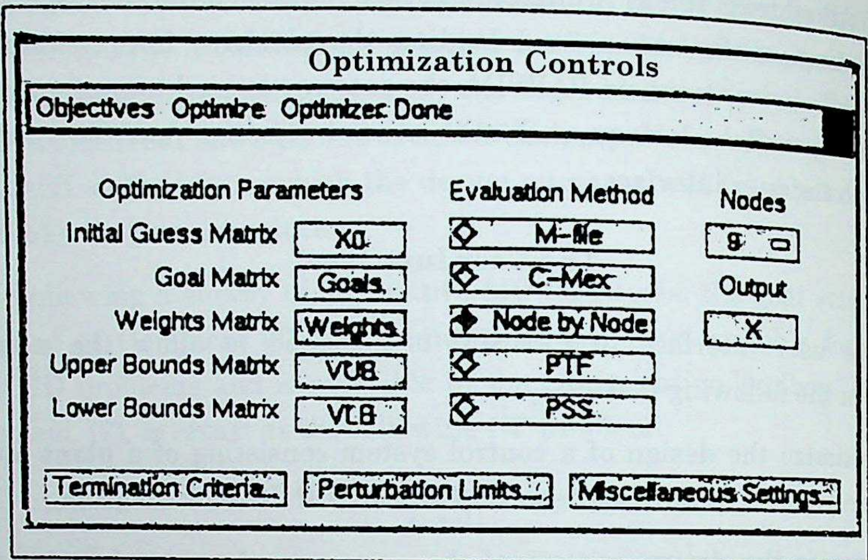


**Fig. 2.** PARSIM optimization interface.

The text inputs in the center of the screen (Fig. 2) are designed to assist the control engineer in formulating the optimization problem by the selection of the optimization parameters. These parameters may be entered numerically or by the use of variable names from the MATLAB workspace. In the center column of the window are a set of mutually exclusive choice buttons for determining the evaluation method of the system model at each epoch of the optimization. Objective function evaluation may be achieved through MATLAB function calls (*M*-file), to dynamically linked C code routines (C-Mex), or by calls to the *parallel platform*. Three parallel methods are supported, node-by-node, parallel transfer functions (PTF), and parallel state space (PSS). These parallel methods are described in more detail in the following section. When a parallel evaluation method is chosen, a further button is used to select the number of processors to be used. A further text input is supplied for selecting the destination variable for results of the optimization. The three buttons at the bottom of the window are used to set parameters of the optimization routine such as the number of interactions allowed, the perturbation in the search space, and the termination criteria. The optimization shown in Fig. 2 is equivalent to the MATLAB command line:

X = attgoal ('goalfun', X0, Goals, Weights, VLB, VUB);

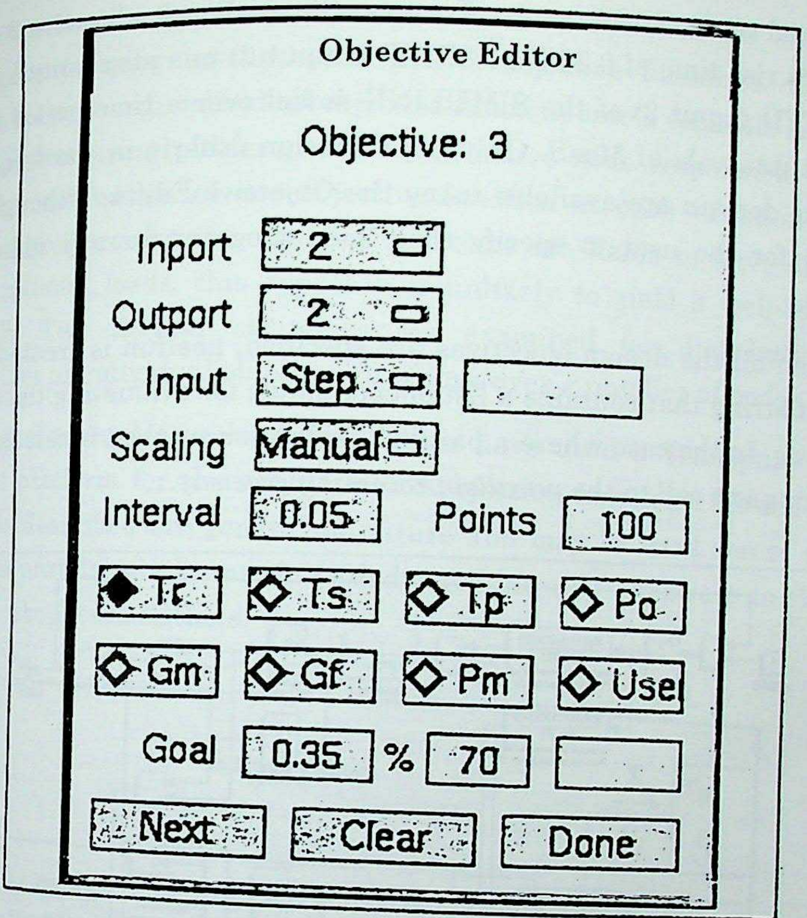


**Fig. 3.** The objective editor.

where X0 is the starting point for system variables, X is the result of the optimization, and VLB and VUB are upper and lower bounds on the value of the solution such that $VLB \leq X \leq VUB$. Goals and Weights are the desired design goals and corresponding weighting factors and goalfun is the vector of objective functions. Goals and Weights may be entered directly as a matrix into the optimization control window or derived automatically from the objective editor. The routine "goalfun" is produced on invocation of the optimization routine by examining the current set of objectives and the evaluation method required.

Consider the SIMULINK model of a two-input, two-output gas turbine engine shown in Fig. 4. The objective editor, Fig. 3, is designed to allow the control engineer to specify and display the design goals in a

simple and convenient manner. For example, in Fig. 3, objective 3 is set for a 70% rise time of 0.35 S at XDIFF (output 2) to a step input applied to XDIFFD (input 2) of the SIMULINK model over a time period 0.0 to 5.0 S at intervals of 50mS. Only direct design criteria in the frequency and time domain are available using the Objective Editor, although it is possible for the user to specify their own program for any individual objective.

When all the design objectives are specified, goalfun is created. This is a text string that contains a list of commands for evaluating the design objectives. In the case where a parallel evaluation method is selected, the text string is a call to the *parallel processing gateway*.

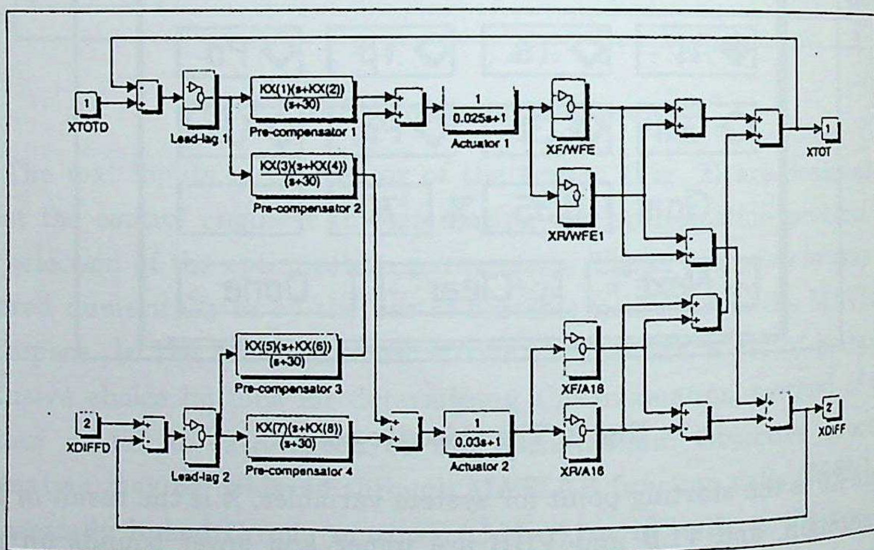


**Fig. 4.** SIMULINK model of a two-input, two-output gas turbine engine.

## Parallel Processing Strategies

Experience has shown that conventional NP algorithms, central to the MO method, are, by and large, unsuited to parallel implementation [7]. In the numerical solution of the MO of equation (3), the set of constraints must be tested at each iteration of the optimization process. In practice, the evaluation of the objective functions at each iteration of the optimization cycle will often comprise the major computational load of the optimization.

Here, we discuss three strategies used by PARSIM for employing parallel processing in the evaluation of objective functions. The first strategy maps each individual objective function onto a separate node in a parallel processing platform, with each active node responsible for the complete evaluation of one objective function at each iteration of the optimization process. As objective functions are liable to have different computational loads, this approach is unlikely to yield a well-balanced solution. Two further strategies are examined for distributing the evaluation of individual objective functions over a number of nodes.

In the examples presented here, we have used a Transputer-based parallel platform for the evaluation of objective functions. However, the methods described are general in nature and may be used across a wide range of parallel platforms, from dedicated parallel hardware to clusters of networked workstations.

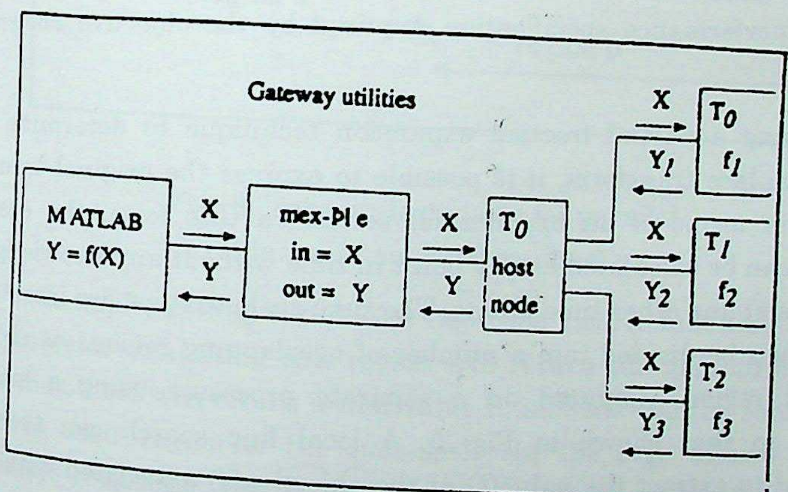


**Fig. 5.** Mapping objective functions onto parallel processor nodes.

## Node-by-Node Parallelism

The most basic method of exploiting parallelism in the MO processes is to map each objective evaluation onto a separate processor [1]. The example in Fig. 5 shows a family of three objectives, $\underline{Y} = \underline{f}(\underline{X})$, mapped onto three compute nodes.

From within PARSIM the three objectives are generated using the Objective Editor. At each function evaluation point of the optimization, a call is made to the parallel processing gateway with the current estimate of the values of the free variables contained in $X$. The gateway passes these values to a hot node on the parallel platform $(T_0)$, which in turn distributes $X$ to the active compute nodes ($T_0$, $T_1$, and $T_2$) and collects the results ($Y$) for return to the optimization routine. Note that the host node is also used as a compute node in this example.

## Parallel Transfer Functions

The parallel transfer function method extends upon the node-by-node parallelism by evaluating a single simulation and all related objectives over a number of nodes [1]. For example, consider the XDIFF/XDIFFD step response, shown in Fig. 6, of the gas turbine engine and the 70% rise-time performance specification required by the objective shown in Fig. 3.

By using a partial fraction expansion technique to determine the inverse Laplace transform, it is possible to express the original transfer function in terms of an exponential series. In this form, the system response can be calculated at any point in time without any knowledge of the values at any other time points. The time-scale that we are interested in may then be divided into a number of overlapping regions as shown, and each region computed on a separate processor using a similar mapping to that shown in Fig. 5. A local line search can then be attempted to extract the value(s) of the objective(s) associated with this simulation.

Overlapping regions are used to ensure that in cases where the solution point lies on the boundary of two regions, at least one processor will successfully compute the solution point. Also, as the line search is generally significantly less computationally expensive than the system simulation, it may be beneficial to perform searches for more than one objective on each node.

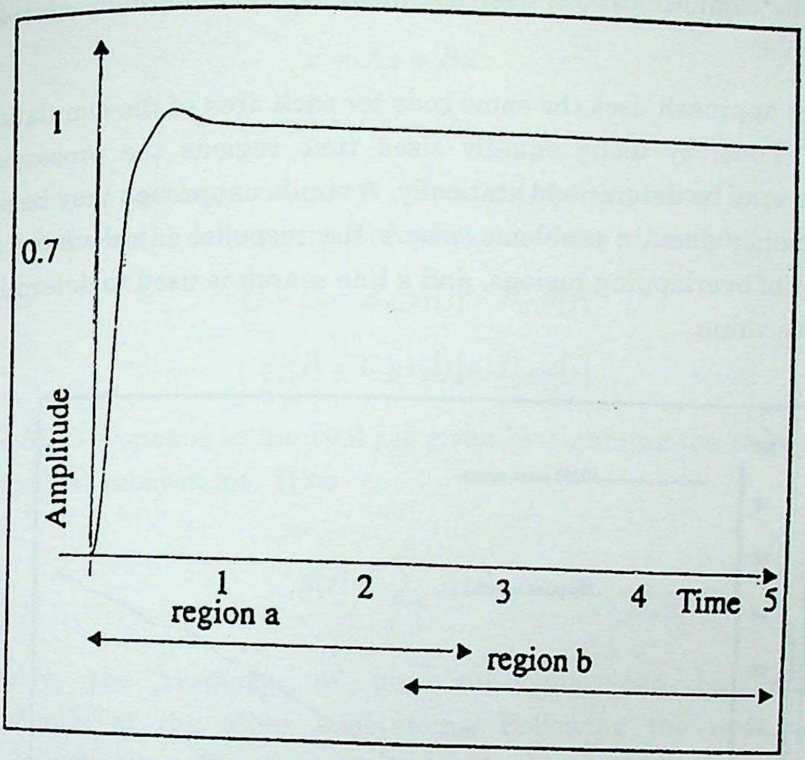


**Fig. 6.** Time-scale division for parallel evaluation of an objective function.

For example, in Fig. 6 the 70% rise time clearly falls into region a. The processor computing this region will return the appropriate value, while the others will return a null value to indicate that no solution was found in their region of the time-scale. In the case of two or more processors returning "active" values, when, say, the value falls into the overlap between two regions, the host node arbitrates and returns only the correct value to MATLAB.

Fig. 7 shows speed-up ratios for varying granularity for a step response of the gas turbine engine. As the size of the individual node's search space is reduced and the number of processors employed increased, the familiar trade-off between communications and computation is encountered. With a relatively large number of search points (10,000), the computation time is dominant and a near linear speed-up is achieved. When the total number of search points is small

(100), the communications overhead dominate and virtually no speed-up is possible.

This approach uses the same code for each area of the simulation and search. Thus, by using equally sized time regions the processor load balance may be determined statically. A similar approach may be applied to frequency domain problems, where the response is calculated over a number of overlapping regions, and a line search is used to determine the objective value.

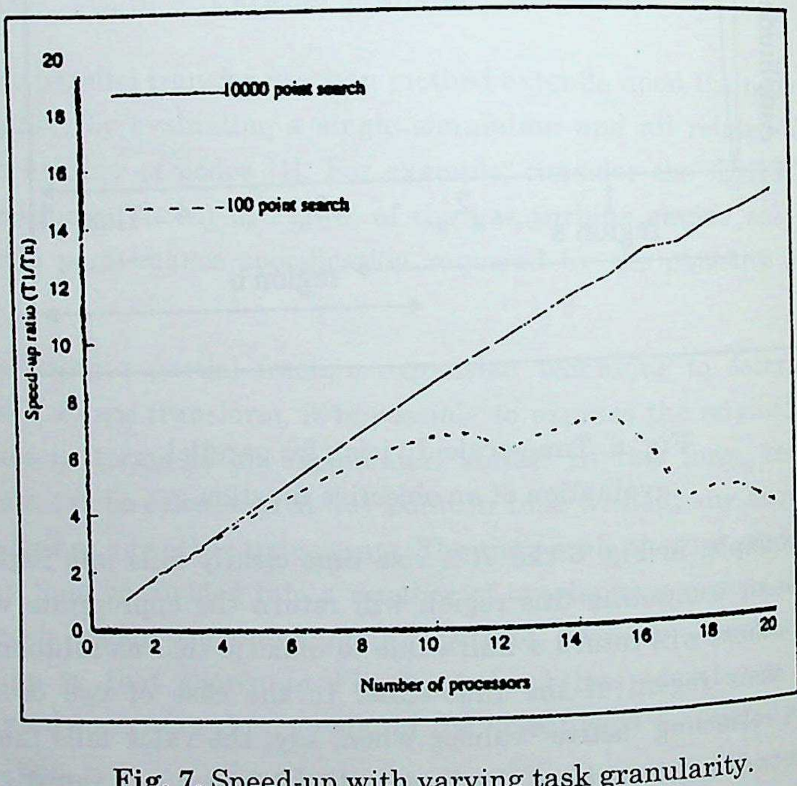


**Fig. 7.** Speed-up with varying task granularity.

## Parallel State Space

The parallel state space method [3] involves decomposing a (high-order) state space model into a number of first- and second-order subsystems derived from the parallel Jordan canonical form of the original system. This not only exposes a level of parallelism that may be exploited when simulating the system, but has the added side effect of reducing the overall numerical complexity of the system simulation.

Consider the continuous SISO state space system:

$$x = Ax + Bu$$

$$y = Cx + Du. \tag{5}$$

By converting this to discrete time and decomposing into parallel state space form, the following set of subsystems are achieved:

$$x_i[j+1] = A_{di}x_i[j] + B_{di}u[j]$$

$$y_i[j] = C_{di}x_i[j] + D_{di}u[j], \tag{6}$$

and the SISO response at interval $j$ is given by summing the responses of the individual subsystems. Thus

$$y[j] = \sum_{i=1}^{n} y_i[j]. \tag{7}$$

Clearly, the response of any subsystem can be computed independently of the other subsystems. Following the node-by-node approach outlined in Fig. 5, the individual tasks are distributed among the selected nodes on the parallel platform by the host node. The host node also has the task of performing the final summation of the individual response vectors, $y_i$ and line search(es) to determine the value of the objective(s).

Consider again the objective defined in Fig. 3. The speed-up performance of the parallel state space method for this objective, using a processor star topology with the subsystem evaluation distributed evenly over the available nodes, is shown in Fig. 8 for a simulation length of 400 time intervals.

Other design objectives and computational methods may easily be incorporated into this framework. For example, the calculation of the multivariable frequency response matrix and the solution of the algebraic Riccati equation may be efficiently computed on distributed memory parallel machines [6]. Such methods will be addressed in future versions of PARSIM.

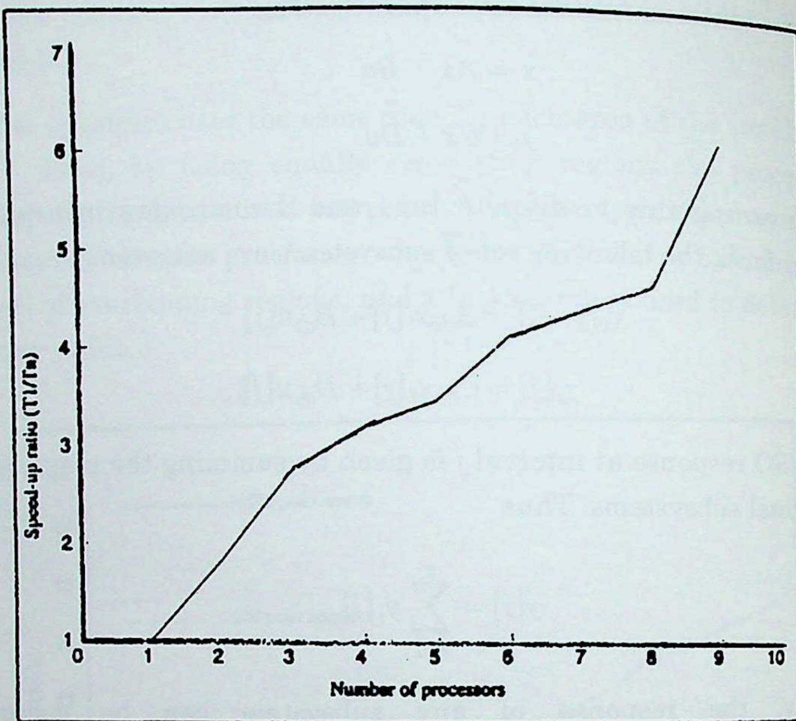


**Fig. 8.** Speed-up for the gas turbine engine simulated over 400 time points.

## Concluding Remarks

Multiobjective optimization has proved to be a powerful tool in the design of control systems attempting to satisfy a set of competing design objectives. The user interface provides a convenient mechanism for directing the optimization during the course of a design. Individual objectives may be changed and the trade-offs between them evaluated in terms of the original design specification, thus improving the level of designer/design tool interaction. Using this framework, it has been possible to alleviate the computational burden by accessing a parallel platform, reducing the optimization design cycle time, and further increasing user interaction. This has the benefit of either allowing more design options to be considered in the same time as the sequential design process or reducing the overall design cycle time.

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# WEIGHTED $H^1(R^n)$ ESTIMATES FOR COMMUTATORS OF SINGULAR INTEGRAL OPERATORS

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## Abstract

We show that the commutator of Coifman, Rochberg and Weiss is a bounded operator from weighted Block $H^1$ space to weighted weak $L^1$ space.

## 1. Introduction

Let $b$ be a locally integrable function on $R^n$ and let $T$ be a Calderón-Zygmund singular integral operator (see Section 2). Consider the commutator operator $[b, T]$ defined by

$$[b, T]f = b \cdot Tf - T(bf).$$

Coifman, Rochberg and Weiss [2] proved that $[b, T]$ is a bounded operator on $L^p(R^n)$, $1 < p < \infty$, when $b$ is a BMO function. Alvarez, Bagby, Kurtz and Pérez [1] proved that $[b, T]$ is a bounded operator on weighted $L^p_w(R^n)$, $1 < p < \infty$, when $b$ is a BMO function.

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But this operator is not weak type (1, 1) (see [5, p. 175]). Pérez [5] and the author [4] defined some variants of $H^1(R^n)$ and Pérez [5] obtained that $[b, T]$ is a bounded operator from $H_b^1(R^n)$ to $L^1(R^n)$ and Komori [4] obtained that $[b, T]$ is a bounded operator from $H_B^1(R^n)$ to weak $L^1(R^n)$.

In this paper we show that $[b, T]$ is a bounded operator from weighted $H_{b,w}^1(R^n)$ to weighted $L_w^1(R^n)$ and $[b, T]$ is a bounded operator from weighted $H_{B,w}^1(R^n)$ to weighted weak $L_w^1(R^n)$.

## 2. Definitions and Theorems

We use the following notation. For a measurable set $E \subset R^n$ and locally integrable function $w$, we write $|E| = \int_E dx$ and $w(E) = \int_E w(x)\,dx$. We write a ball of radius $r$ and centered at $x_0$ by $Q = Q(x_0, r)$.

Throughout this paper, we always use the letter $C$ to denote a positive constant that may vary at occurrence but is independent of the essential variables.

**Definition 1.** We say that $T$ is a *Calderón-Zygmund singular integral operator* if $T$ satisfies the following conditions:

$$T \text{ is a bounded operator on } L^2(R^n). \tag{1}$$

$$Tf(x) = p.v. \int_{R^n} K(x-y)\,f(y)\,dy. \tag{2}$$

$$|K(x)| \le \frac{C}{|x|^n}, \quad x \ne 0. \tag{3}$$

$$|K(x-y) - K(x)| \le \frac{C|y|}{|x|^{n+1}}, \quad |x| > 2|y|. \tag{4}$$

**Definition 2.** For a nonnegative locally integrable function $w$, we say that $w \in A_1$ if $w$ satisfies the following:

$$\frac{w(Q)}{|Q|} \leq C \text{ ess } \inf_{x \in Q} w(x) \qquad \text{for all balls } Q.$$

**Definition 3.** For a nonnegative locally integrable function $w$, we say that $w \in RH_q$, $q \geq 1$ (reverse Hölder condition), if $w$ satisfies the following:

$$\left(\frac{1}{|Q|}\int_Q w(x)^q\,dx\right)^{1/q} \leq \frac{C}{|Q|}\int_Q w(x)\,dx \qquad \text{for all balls } Q.$$

**Definition 4.** Let $w$ be a nonnegative locally integrable function. We say that a function $f$ belongs to $L^1_w(R^n)$ if $f$ satisfies

$$\|f\|_{L^1_w} = \int_{R^n} |f(x)|\,w(x)\,dx < \infty.$$

**Definition 5.** Let $w$ be a nonnegative locally integrable function. We say that a function $f$ belongs to $L^{1,\infty}_w(R^n)$ if $f$ satisfies

$$\|f\|_{L^{1,\infty}_w} = \sup_{\lambda > 0} \lambda \cdot w(\{x \in R^n; |f(x)| > \lambda\}) < \infty.$$

**Definition 6.** We say that a locally integrable function $b$ belongs to $\mathrm{BMO}(R^n)$ if $b$ satisfies the following:

$$\|b\|_* = \sup_Q \frac{1}{|Q|}\int_Q |b(x) - b_Q|\,dx < \infty,$$

where $b_Q = |Q|^{-1}\int_Q b(x)\,dx$ and the supremum is taken over all balls.

The usual weighted Hardy space is defined as follows ([6, p. 111]).

**Definition 7.** Let $w \in A_1$. A function $a$ is a *w-atom* if there exists a ball $Q$ such that the following conditions are satisfied

$$\operatorname{supp}(a) \subset Q, \qquad \text{(i)}$$

$$\| a \|_{L^\infty} \le w(Q)^{-1}, \tag{ii}$$

$$\int_Q a(x)\,dx = 0. \tag{iii}$$

The weighted Hardy space $H_w^1(R^n)$ consists of the subspace of $L_w^1(R^n)$ of functions $f$ which can be written as $f = \sum_j c_j a_j$, where $a_j$ are $w$-atoms and $c_j$ are real numbers with $\sum_j | c_j | < \infty$.

Following Pérez [5], we define $H_{b,\,w}^1$ space.

**Definition 8.** Let $w \in A_1$. A function $a$ is a $(b, w)$*-atom* if there exists a ball $Q$ such that the following conditions are satisfied (i), (ii), (iii) and

$$\int_Q a(x)\,b(x)\,dx = 0. \tag{iv}$$

The space $H_{b,\,w}^1(R^n)$ consists of the subspace of $L_w^1(R^n)$ of functions $f$ which can be written as $f = \sum_j c_j a_j$, where $a_j$ are $(b, w)$-atoms and $c_j$ are real numbers with $\sum_j | c_j | < \infty$.

We define $H_{B,\,w}^1$ (the weighted Block $H^1$ space) as follows:

**Definition 9.** Let $w \in A_1$. We say that a function $f$ belongs to $H_{B,\,w}^1(R^n)$ if $f$ can be written as

$$f = \sum_{j=1}^{\infty} c_j a_j, \tag{5}$$

where $a_j$ are $w$-atoms (see Definition 7) and $c_j$ are real numbers with

$$\sum_{j=1}^{\infty} | c_j | \left( 1 + \log^+ \frac{1}{| c_j |} \right) < \infty.$$

And let $\| f \|_{H^1_{B,\,w}}$ be the infimum of $\sum_{j=1}^{\infty} | c_j | \left(1 + \log\left(\left(\sum_i | c_i |\right) \Big/ | c_j |\right)\right)$ over all representations (5) of $f$.

We obtain the following theorems.

**Theorem 1.** *Let $T$ be a Calderón-Zygmund singular integral operator and let $b$ be a function in $BMO(R^n)$ and $w \in A_1$. Then $[b, T]$ is a bounded operator from $H^1_{b,\,w}(R^n)$ to $L^1_w(R^n)$.*

**Theorem 2.** *Let $T$ be a Calderón-Zygmund singular integral operator and let $b$ be a function in $BMO(R^n)$ and $w \in A_1$. Then $[b, T]$ is a bounded operator from $H^1_{B,\,w}(R^n)$ to $L^{1,\,\infty}_w(R^n)$ and*

$$\| [b, T] f \|_{L^{1,\,\infty}_w} \leq C \| b \|_* \| f \|_{H^1_{B,\,w}}.$$

**Remark.** For $w \equiv 1$, Theorem 1 was obtained by Pérez [5] and Theorem 2 was obtained by Komori [4].

## 3. Lemmas

First we list some elementary lemmas for weight functions without proof (see [3] or [8]).

**Lemma 1** ([8, p. 226]). *If $w \in A_1$, then $w$ satisfies the following doubling condition*

$$w(2Q) \leq Cw(Q), \quad \textit{for all balls } Q$$

*where $2Q$ is the ball with the same center as $Q$ and whose radius is twice as long.*

**Lemma 2** ([8, p. 230]). *If $w \in A_1$, then there exists $q > 1$ such that $w \in RH_q$.*

**Lemma 3** ([8, p. 334]). *Let $T$ be a Calderón-Zygmund singular integral operator and $w \in A_1$. Then $T$ is bounded from $L^1_w(R^n)$ to $L^{1,\,\infty}_w(R^n)$.*

Next lemmas are consequences of the definition of BMO.

**Lemma 4** ([8, p. 203]). *Let $b \in BMO(R^n)$. If $p \geq 1$ and $Q$ is any ball, then*

$$\left(\frac{1}{|Q|}\int_Q |b(x) - b_Q|^p\,dx\right)^{1/p} \leq C_p\|b\|_*,$$

*where $C_p$ is a constant depending only on $p$ and $n$.*

**Lemma 5** ([8, p. 206]). *Let $b \in BMO(R^n)$, $Q = Q(x_0, r)$ be the ball of radius $r$ and centered at $x_0$ and $Q_j = Q(x_0, 2^j r)$, where $j = 1, 2, 3, \dots$. Then*

$$\left|b_Q - b_{Q_j}\right| \leq Cj\|b\|_*.$$

**Lemma 6.** *Let $b \in BMO(R^n)$ and $w \in A_1$. Then for any ball $Q = Q(x_0, r)$,*

$$\int_{\complement 2Q} \frac{|b(x) - b_Q|}{|x - x_0|^{n+1}}\,w(x)\,dx \leq \frac{Cw(Q)\|b\|_*}{r|Q|}.$$

**Proof.** By Lemma 2, we may assume $w \in RH_q$ for some $q > 1$. Let $Q_j = Q(x_0, 2^j r)$. Then

$$\int_{\complement 2Q} \frac{|b(x) - b_Q|}{|x - x_0|^{n+1}}\,w(x)\,dx$$

$$\leq \sum_{j=1}^{\infty}\left\{\int_{2^j r \leq |x - x_0| < 2^{(j+1)} r} \frac{\left|b(x) - b_{Q_{j+1}}\right|}{|x - x_0|^{n+1}}\,w(x)\,dx\right.$$

$$\left. + \left|b_Q - b_{Q_{j+1}}\right| \int_{2^j r \leq |x - x_0| < 2^{(j+1)} r} \frac{w(x)\,dx}{|x - x_0|^{n+1}}\right\}$$

$$= \sum_{j=1}^{\infty}\{I_j + II_j\}.$$

First we estimate $I_j$.

$$I_j \le \frac{C}{2^j r} \frac{1}{|Q_{j+1}|} \int_{Q_{j+1}} \left| b(x) - b_{Q_{j+1}} \right| w(x)\, dx$$

$$\le \frac{C}{2^j r} \left( \frac{1}{|Q_{j+1}|} \int_{Q_{j+1}} \left| b(x) - b_{Q_{j+1}} \right|^p dx \right)^{1/p} \left( \frac{1}{|Q_{j+1}|} \int_{Q_{j+1}} w(x)^q\, dx \right)^{1/q},$$

where $1/p + 1/q = 1$.

By Lemma 4, we have

$$\left( \frac{1}{|Q_{j+1}|} \int_{Q_{j+1}} \left| b(x) - b_{Q_{j+1}} \right|^p dx \right)^{1/p} \le C \| b \|_*.$$

Because $w \in A_1$ and $w \in RH_q$, we have

$$\left( \frac{1}{|Q_{j+1}|} \int_{Q_{j+1}} w(x)^q\, dx \right)^{1/q} \le \frac{C}{|Q_{j+1}|} \int_{Q_{j+1}} w(x)\, dx \le C \operatorname*{ess\,inf}_{x \in Q} w(x)$$

and

$$\left( \frac{1}{|Q_{j+1}|} \int_{Q_{j+1}} w(x)^q\, dx \right)^{1/q} \le \frac{C w(Q)}{|Q|}.$$

So, we obtain

$$I_j \le \frac{C 2^{-j} w(Q) \| b \|_*}{r |Q|}.$$

Next, we estimate $II_j$. By the same argument as above and Lemma 5, we have

$$II_j \le \frac{C j \cdot 2^{-j} w(Q) \| b \|_*}{r |Q|}.$$

Thus we obtain the desired result.

The following lemma is most essential to prove Theorem 2.

**Lemma 7** ([7]). *Let* $w \geq 0$. *If* $\{f_j\}$ *is a sequence of measurable functions satisfying* $\| f_j \|_{L^{1,\infty}_w} \leq 1$, *and* $\{c_j\}$ *is a numerical sequence, then*

$$\left\| \sum_{j=1}^{\infty} c_j f_j(x) \right\|_{L^{1,\infty}_w} \leq C \sum_{j=1}^{\infty} | c_j | \left( 1 + \log \frac{\sum_{i=1}^{\infty} | c_i |}{| c_j |} \right).$$

**Remark.** Stein and Weiss [7] obtained above result for $w \equiv 1$. By the same argument, we can easily obtain this lemma.

## 4. Proof

The proofs of two theorems are similar, so we prove only Theorem 2.

**Proof of Theorem 2.** By Lemma 2, we may assume $w \in RH_q$ for some $q > 1$.

By Lemma 7, it is enough to show that there exists a constant $C$ such that

$$\| [b, T] a \|_{L^{1,\infty}_w} \leq C \| b \|_* \quad \text{for each } w\text{-atom } a.$$

To prove this, suppose that $\text{supp}(a) \subset Q$ for some ball $Q = Q(x_0, r)$ (center $x_0$ and radius $r$). We write

$$[b, T] a(x) = [b, T] a(x) \chi_{2Q}(x) + [b, T] a(x) \chi_{\complement 2Q}(x) = I + II,$$

where $\chi_E$ is a characteristic function of a set $E$.

We shall show that $I$ belongs to $L^1_w(R^n)$ and $II$ belongs to $L^{1,\infty}_w(R^n)$.

The estimate for $I$ follows from the boundedness of $[b, T]$ on $L^2_w(R^n)$ ([1]). We obtain

$$\| I \|_{L^1_w} \leq w(2Q)^{1/2} \| [b, T] a \|_{L^2_w} \leq C w(Q)^{1/2} \| b \|_* \| a \|_{L^2_w} \leq C \| b \|_*$$

because $w$ satisfies *doubling condition* (Lemma 1).

To estimate $II$, we write

$$II = (b(x) - b_Q)\,Ta(x) \cdot \chi_{\complement 2Q}(x) - T((b - b_Q)\,a)(x) \cdot \chi_{\complement 2Q}(x) = III + IV.$$

We have

$$\int_{\complement 2Q} |\, III \,|\, w(x)\, dx \le \frac{C|\,Q\,|}{w(Q)} \int_{\complement 2Q} \frac{r \cdot |\, b(x) - b_Q \,|}{|\, x - x_0 \,|^{n+1}}\, w(x)\, dx \le C \,\|\, b \,\|_*$$

by Lemma 6.

Because $T$ is bounded from $L_w^1$ to $L_w^{1,\,\infty}$ (Lemma 3), we have

$$\|\, IV \,\|_{L_w^{1,\,\infty}} \le C \big\|\, (b - b_Q)\, a \,\big\|_{L_w^1} \le \frac{C}{w(Q)} \int_Q |\, b(x) - b_Q \,|\, w(x)\, dx$$

$$\le \frac{C}{w(Q)} \left( \int_Q |\, b(x) - b_Q \,|^p\, dx \right)^{1/p} \left( \int_Q w(x)^q\, dx \right)^{1/q},$$

where $1/p + 1/q = 1$.

Because $w \in RH_q$, we have

$$\frac{1}{w(Q)} \left( \int_Q w(x)^q\, dx \right)^{1/q} \le C|\, Q \,|^{-1/p}.$$

So the last term is bounded by

$$C \left( \frac{1}{|\, Q \,|} \int_Q |\, b(x) - b_Q \,|^p\, dx \right)^{1/p} \le C \|\, b \,\|_*$$

because of Lemma 4.

Thus we obtain the result.

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# ON $g$-ALMOST REGULAR SPACES

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## Abstract

The aim of this paper is to define $g$-almost regular spaces, $rg$-continuous functions and $r$-open functions and use these functions to study some preservation theorems for $g$-almost regularity.

## 1. Introduction

In 1970, Levine [2] introduced $g$-closed sets in topological spaces. In 1980, Munchi [5] introduced the notion of $g$-regular spaces by replacing closed sets in the definition of regular spaces by $g$-closed sets. In this paper the separation axiom known as the $g$-almost regular is defined and the necessary and sufficient conditions for a space $X$ to be $g$-almost regular is studied. Also $rg$-continuous functions and $r$-open functions are defined and used to study some preservation theorems for $g$-almost regularity.

Throughout this paper, spaces $X$ and $Y$ mean topological spaces. For a subset $A$ of $X$, the closure of $A$ and the interior of $A$ are denoted by $Cl(A)$ and $Int(A)$, respectively. $A$ is said to be *regular open* (resp. *regular closed*) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). The family of regular open (resp. regular closed) sets of a space $X$ is denoted by $RO(X)$

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(resp. $RC(X)$).

## 2. Definitions and Basic Properties

**Definition 1.** A subset $A$ of $X$ is said to be

(a) *generalized closed* [2], (briefly $g$-closed) if $Cl(A) \subset U$ whenever $A \subset U$ and $U$ is open in $X$.

(b) *regularly generalized closed* [3], (briefly $rg$-closed) if $Cl(A) \subset U$ whenever $A \subset U$ and $U \in RO(X)$.

**Remark.** It is obvious that if $A$ is closed it is $g$-closed and therefore it is $rg$-closed.

**Definition 2.** A space $X$ is said to be *generalized almost regular* (briefly $g$-almost regular) if for each $rg$-closed set $A$ of $X$ and each $x \in X - A$ there exist disjoint open sets $U$ and $V$ of $X$ such that $A \subset U$ and $x \in V$.

**Remark.** If $X$ is $g$-almost regular, then $X$ is almost regular but the converse may not be true as is seen in the following example.

**Example 3.** Let $X = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}, \{b, c\}\}$. Then $X$ is almost regular but it is not $g$-almost regular, since there exists no disjoint open set containing $b$ and $\{c\}$ where $\{c\}$ is $g$-closed set in $X$.

**Theorem 4.** *A space $X$ is $g$-almost regular if and only if for every $rg$-closed set $F$ of $X$ and each $x \in X - F$, there exist open sets $U$ and $V$ such that $x \in U$ and $F \subset V$ and $Cl(U) \cap Cl(V) = \varnothing$.*

**Proof.** Let $X$ be $g$-almost regular, $F$ be $rg$-closed set, $x \in X - F$, so there exist open sets $U_1$ and $V_1$ such that $x \in U_1$, $F \subset V_1$ and $U_1 \cap V_1 = \varnothing$, implies that $U_1 \cap Cl(V_1) = \varnothing$. Since $Cl(V_1)$ is closed, it is $rg$-closed. Also $x \in X - Cl(V_1)$, so there exist open sets $U_2$ and $V_2$ such that $x \in U_2$, $Cl(V_1) \subset V_2$ and $U_2 \cap V_2 = \varnothing$. Therefore, $Cl(U_2) \cap V_2 = \varnothing$. Put $U = U_1 \cap U_2$, so $U$ and $V_1$ are open sets of $X$, $x \in U$, $F \subset V_1$ and $Cl(U) \cap Cl(V_1) = \varnothing$.

Sufficiency is obvious.

## 3. Preservation Theorems

**Definition 5.** A function $f : X \to Y$ is said to be

(a) *almost-open* [7], if $f(U)$ is open in $Y$ for every $U \in RO(X)$.

(b) *regularly generalized closed* [6], (briefly $rg$-closed) if for each closed set $F$ of $X$, $f(F)$ is $rg$-closed.

(c) *generalized closed* [4], (briefly $g$-closed) if for each closed set $F$ of $X$, $f(F)$ is $g$-closed.

(d) *generalized continuous* [1], (briefly $g$-cts) if $f^{-1}(F)$ is $g$-closed in $X$ for every closed set $F$ of $Y$.

(e) *generalized closed irresolute* [1], (briefly $gc$-irresolute) if $f^{-1}(F)$ is $g$-closed in $X$ for every $g$-closed set $F$ of $Y$.

(f) *regularly generalized continuous* (briefly $rg$-cts) if $f^{-1}(F)$ is $rg$-closed for every closed set $F$ of $Y$.

(g) *regularly open* (briefly $r$-open) if $f(V) \in RO(Y)$ whenever $V$ is open in $X$.

**Lemma 6.** *If $f : X \to Y$ is r-open, rg-cts bijective function and $F$ is rg-closed subset of $Y$, then $f^{-1}(F)$ is rg-closed in $X$.*

**Proof.** Let $F$ be $rg$-closed in $Y$ and $f^{-1}(F) \subset V$ where $V \in RO(X)$. Then $f(f^{-1}(F)) \subset f(V)$ and $F \subset f(V)$. Since $f$ is $r$-open function and $V$ is open, $f(V) \in RO(Y)$. Also $F$ is $rg$-closed, $Cl(F) \subset f(V)$, since $Cl(F)$ is closed in $Y$ and $f$ is $rg$-cts, $f^{-1}(Cl(F))$ is $rg$-closed in $X$, but $f^{-1}(Cl(F)) \subset f^{-1}(f(V)) = V$, so $Cl(f^{-1}(Cl(F))) \subset V$, implies that $Cl(f^{-1}(F)) \subset Cl(f^{-1}(Cl(F))) \subset V$, and $f^{-1}(F)$ is $rg$-closed in $X$.

**Theorem 7.** *If $f : X \to Y$ is r-open rg-cts bijective function and X is g-almost regular, then Y is g-almost regular.*

**Proof.** Let $F$ be $rg$-closed set in $Y$ and $y \in Y - F$, since $f$ is $r$-open and $rg$-cts bijective, $f^{-1}(F)$ is $rg$-closed in $X$. Let $f(x) = y$, so $x \in X - f^{-1}(F)$ but $X$ is $g$-almost regular, so there exist open sets $U$ and $V$ such that $x \in U$, $f^{-1}(F) \subset V$ and $U \cap V = \emptyset$, hence $y = f(x) \in f(U)$, $F \subset f(V)$. Since $f$ is $r$-open, $f(U) \in RO(Y)$ and $f(V) \in RO(Y)$, so they are open and $f(U) \cap f(V) = \emptyset$. Hence $Y$ is $g$-almost regular.

**Lemma 8.** *If $f : X \to Y$ is rg-cts, rg-closed function and F is rg-closed set in X, then $f(F)$ is rg-closed in Y.*

**Proof.** Let $F$ be $rg$-closed set in $X$ and $f(F) \subset V \in RO(Y)$. Since $f$ is $rg$-cts, $f^{-1}(V) \in RO(X)$. But $F$ is $rg$-closed and therefore $Cl(F) \subset f^{-1}(V)$ and $f(Cl(F)) \subset V$. Since $Cl(F)$ is closed in $X$ and $f$ is $rg$-closed, $f(Cl(F))$ is $rg$-closed in $Y$ and $f(Cl(F)) \subset V$, so $Cl(f(Cl(F))) \subset V$ and $Cl(f(F)) \subset V$. Hence $f(F)$ is $rg$-closed in $Y$.

**Theorem 9.** *If $f : X \to Y$ is rg-cts, rg-closed injective function and Y is g-almost regular, then X is g-almost regular.*

**Proof.** Let $F$ be $rg$-closed in $X$ and $x \in X - F$. From Lemma 8, $f(F)$ is $rg$-closed in $Y$ and $f(x) \in Y - f(F)$. Since $Y$ is $g$-almost regular, there exist open sets $U$ and $V$ such that $f(x) \in U$, $F \subset V$ and $U \cap V = \emptyset$, so $x \in f^{-1}(U)$ is open, $F \subset f^{-1}(V)$ which is open and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ implies that $X$ is $g$-almost regular.

**Definition 10.** A space $X$ is said to be *g-regular* [5], if for each $g$-closed set $F$ of $X$ and each point $x \in X - F$, there exist disjoint open sets $U$ and $V$ of $X$ such that $x \in U$ and $F \subset V$.

**Theorem 11.** *If $f : X \to Y$ is open, g-cts bijective function and X is g-almost regular, then Y is g-regular.*

**Proof.** Let $F$ be $g$-closed in $Y$, $y \in Y - F$. Since $f$ is open and $g$-cts function it is $gc$-irresolute [1], so $f^{-1}(F)$ is $g$-closed in $X$, therefore it is $rg$-closed. Let $f(x) = y$, $x \in X - f^{-1}(F)$. Since $X$ is $g$-almost regular, there exist open sets $U$ and $V$ such that $x \in U$, $f^{-1}(F) \subset V$ and $U \cap V = \varnothing$. Since $f$ is open bijective, we have $y \in f(U)$, $F \subset f(V)$ and $f(U) \cap f(V) = \varnothing$, therefore $Y$ is $g$-regular.

**Lemma 12.** *If* $f : X \to Y$ *is r-open, g-cts bijective function and F is rg-closed set in Y, then* $f^{-1}(F)$ *is g-closed in X.*

**Proof.** Let $F$ be $rg$-closed in $Y$ and $V$ be open in $X$ such that $f^{-1}(F) \subset V$, so $ff^{-1}(F) \subset f(V)$. Since $f$ is $r$-open bijective, $F \subset f(V) \in RO(Y)$, but $F$ is $rg$-closed, then $Cl(F) \subset f(V)$. Since $Cl(F)$ is closed in $Y$ and $f$ is $g$-cts, $f^{-1}(Cl(F))$ is $g$-closed in $X$ and $f^{-1}(Cl(F)) \subset V$ is open, so $Cl(f^{-1}(Cl(F))) \subset V$, but $Cl(f^{-1}(F)) \subset Cl(f^{-1}(Cl(F))) \subset V$, and hence $f^{-1}(F)$ is $g$-closed in $X$.

**Theorem 13.** *If* $f : X \to Y$ *is r-open, g-cts bijective function and X is g-regular, then Y is g-almost regular.*

**Proof.** Let $F$ be $rg$-closed in $Y$, $y \in Y - F$. From Lemma 12, $f^{-1}(F)$ is $g$-closed in $X$. Put $f(x) = y$, so $x \in X - f^{-1}(F)$. Since $X$ is $g$-regular, there exist open sets $U$ and $V$ such that $x \in U$, $f^{-1}(F) \subset V$ and $U \cap V = \varnothing$. Since $f$ is $r$-open bijective, $F$, $y \in f(U)$, $F \subset f(V)$ and $f(U) \cap f(V) = \varnothing$. Thus $Y$ is $g$-almost regular.

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# ON $q$-IDEALS IN $BCI$-ALGEBRAS

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## Abstract

We give characterizations of $q$-ideals and $a$-ideals, respectively. Moreover, we find relations among $q$-ideals, $a$-ideals and $p$-ideals.

## 1. Introduction

In 1966, Iséki [2] introduced the notion of $BCI$-algebras which is a generalization of $BCK$-algebras. The ideal theory plays an important role in studying $BCK/BCI$-algebras. Liu et al. [5] introduced the notions of $q$-ideals and $a$-ideals in $BCI$-algebras and used them to characterize quasi-associative $BCI$-algebras and associative $BCI$-algebras, respectively.

In this paper, we give characterizations of $q$-ideals and $a$-ideals, respectively. Moreover, we find relations among $q$-ideals, $a$-ideals and $p$-ideals.

## 2. Preliminaries

We review some definitions and properties that will be useful in our results.

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By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following conditions:

(I1) $((x * y) * (x * z)) * (z * y) = 0$,

(I2) $(x * (x * y)) * y = 0$,

(I3) $x * x = 0$,

(I4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A *BCI*-algebra $X$ satisfying $0 * x = 0$ for all $x \in X$ is called a *BCK-algebra*. In any *BCI*-algebra $X$ one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

A *BCI*-algebra $X$ has the following properties for any $x, y, z \in X$:

(1) $x * 0 = x$,

(2) $(x * y) * z = (x * z) * y$,

(3) $(x * z) * (y * z) \leq x * y$,

(4) $0 * (x * y) = (0 * x) * (0 * y)$.

A nonempty subset $I$ of a *BCK*/*BCI*-algebra $X$ is called an *ideal* of $X$ if it satisfies

($b_1$) $0 \in I$,

($b_2$) $x * y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

Throughout this note $X$ always means a *BCI*-algebra unless specific request.

Any ideal $I$ has the property: $y \in I$ and $x \leq y$ imply $x \in I$. An ideal $I$ is *p-ideal* if and only if $0 * (0 * x) \in I$ implies $x \in I$.

A nonempty set $I$ of $X$ is called *q-ideal* if it satisfies ($b_1$) and

($b_3$) $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$.

A nonempty set $I$ of $X$ is called *a-ideal* if it satisfies ($b_1$) and

($b_4$) $(x * z) * (0 * y) \in I$ and $z \in I$ imply $y * x \in I$.

## 3. Main Results

In [5], Liu et al. proved the following lemma.

**Lemma 3.1** [5]. *If I is an ideal of X, then the following are equivalent: for all* $x, y, z \in X$,

(a) *I is a q-ideal of X,*

(b) $x * (0 * y) \in I$ *implies* $x * y \in I$,

(c) $x * (y * z) \in I$ *implies* $(x * y) * z \in I$.

Now, we investigate characterizations of $q$-ideals.

**Theorem 3.2.** *If I is an ideal of X, then the following are equivalent: for all* $x, y, z \in X$,

(a) *I is a q-ideal of X,*

(b) $(x * z) * (0 * y) \in I$ *implies* $(x * z) * y \in I$,

(c) $(x * z) * (0 * y) \in I$ *and* $z \in I$ *imply* $x * y \in I$.

**Proof.** (a) $\Rightarrow$ (b) By Lemma 3.1(b), it is trivial.

(b) $\Rightarrow$ (c) Let $(x * z) * (0 * y) \in I$ and $z \in I$. Then by (2), we have $(x * y) * z = (x * z) * y \in I$. Since $I$ is an ideal, we get $x * y \in I$ and so (c) holds.

(c) $\Rightarrow$ (a) Let $x * (0 * y) \in I$. Then we get $(x * 0) * (0 * y) \in I$. Since $0 \in I$, by (c), we have $x * y \in I$. Hence $I$ is a $q$-ideal. The proof is complete.

We know that every $q$-ideal is an ideal and a subalgebra. But the converse is not true, in general. Now, we consider the following condition:

For any $q$-ideal $I$ of $X$, let the condition be as follows:

(5) $0 * x \in I$ implies $x \in I$.

We know that, in general, a $q$-ideal of $X$ may not satisfy (5).

**Example 3.3.** Let $X = \{0, a, b, c\}$ be a $BCI$-algebra with Cayley table as follows:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 2 | 2 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 2 | 0 | 0 |
| 3 | 3 | 2 | 1 | 0 |

Routine calculations give that $\{0, 2\}$ and $\{0, 1\}$ are $q$-ideals of $X$. We know that $\{0, 2\}$ does not satisfy (5) because $0 * 1 = 0 \in \{0, 2\}$ but $1 \notin \{0, 2\}$. But we see that $\{0, 1\}$ satisfies (5).

In general, every $q$-ideal may not be a $p$-ideal in $BCI$-algebra. In Example 3.3, $\{0, 2\}$ is a $q$-ideal but not a $p$-ideal of $X$ because $0 * (0 * 3) \in \{0, 2\}$ and $3 \notin \{0, 2\}$. But we have the following theorem.

**Theorem 3.4.** *Any $q$-ideal satisfying the condition* (5) *is a $p$-ideal of $X$.*

**Proof.** Let $I$ be a $q$-ideal satisfying the condition (5) and $0 * (0 * x) \in I$. Then, by Lemma 3.1(b), we get $0 * x \in I$. Thus $x \in I$ because of (5). Hence, $I$ is a $p$-ideal of $X$.

In [5], Liu et al. proved the following lemma.

**Lemma 3.5** [5]. *Let $I$ be an ideal of $X$. Then the following are equivalent: for all $x, y, z \in X$,*

(a) *$I$ is an $a$-ideal of $X$,*

(b) *$(x * z) * (0 * y) \in I$ implies $y * (x * z) \in I$,*

(c) *$x * (0 * y) \in I$ implies $y * x \in I$.*

Now, we investigate characterizations of $a$-ideals.

**Theorem 3.6.** *Let $I$ be an ideal of $X$. Then the following are equivalent: for all $x, y, z \in X$,*

(a) *$I$ is an $a$-ideal of $X$,*

(b) $x * (y * z) \in I$ *implies* $z * (x * y) \in I$,

(c) $x * (y * z) \in I$ *and* $y \in I$ *implies* $z * x \in I$.

**Proof.** (a) $\Rightarrow$ (b) Let $x * (y * z) \in I$. Then

$$\begin{aligned}((x * y) * (0 * z)) * (x * (y * z)) &= ((x * y) * (x * (y * z)) * (0 * z)) \\ &\leq ((y * z) * y) * (0 * z) \\ &= (0 * z) * (0 * z) = 0 \in I.\end{aligned}$$

Since $I$ is an ideal, we get $(x * y) * (0 * z) \in I$. By Lemma 3.5(b), we obtain $z * (x * y) \in I$ and so (b) holds.

(b) $\Rightarrow$ (c) Let $x * (y * z) \in I$ and $y \in I$. Then

$$(x * (0 * z)) * (x * (y * z)) \leq (y * z) * (0 * z) \leq y * 0 = y \in I.$$

Since $I$ is an ideal, we have $x * (0 * z) \in I$. By Lemma 3.5(c), we get $z * x \in I$ and so (c) holds.

(c) $\Rightarrow$ (a) It is obvious by Lemma 3.5(c).

We know that every $a$-ideal is $p$-ideal and $q$-ideal, but the converses need not be true [5].

**Theorem 3.7.** *Every $q$-ideal with condition* (5) *is an $a$-ideal.*

**Proof.** Suppose that $(x * z) * (0 * y) \in I$ and $z \in I$. Since $I$ is a $q$-ideal $x * y \in I$ and $0 * (0 * (x * y)) \in I$. Now

$$\begin{aligned}0 * (y * x) &= (0 * y) * (0 * x) \\ &= (0 * (0 * (0 * y))) * (0 * x) \\ &= (0 * (0 * x)) * (0 * (0 * y)) \\ &= 0 * ((0 * x) * (0 * y)) \\ &= 0 * (0 * (x * y)) \in I.\end{aligned}$$

Using (5), we get $y * x \in I$. Hence $I$ is an $a$-ideal of $X$.

But every $p$-ideal with condition (5) may not be an $a$-ideal as the following example shows:

**Example 3.8.** Let $X = \{0, a, b, c\}$ be a $BCI$-algebra with Cayley table as follows:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 3 | 2 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 2 | 0 | 3 |
| 3 | 3 | 3 | 2 | 0 |

It is easy to show that $\{0, 1\}$ is a $p$-ideal and satisfies condition (5), but it is not an $a$-ideal as: $3 * (0 * 2) = 3 * 3 = 0 \in \{0, 1\}$, but $3 * 2 = 2 \notin \{0, 1\}$.

**Corollary 3.9.** *Let $I$ be a $q$-ideal of $X$, then the following are equivalent*:

(a) *$I$ satisfies* (5),

(b) *$I$ is an $a$-ideal of $X$,*

(c) *$I$ is a $p$-ideal of $X$.*

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# LOCAL CONTRIBUTION MATRICES IN SCHWARZSCHILD SPACE-TIME: A SKETCH FOR A COMPLETE MATHEMATICA NOTEBOOK

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## Abstract

This letter represents a step forward from an earlier contribution made by two of us (Borges and Machado [Far East J. Math. Sci. (FJMS) 3(3) (2001), 467-492]), in which a technique based on Finite Elements was introduced in order to treat partial differential equations of Einstein type. Einstein's partial differential equations are, undoubtedly, among the most complicated set of equations appearing in mathematical physics. They are non-linear and of hyperbolic type. For treating such equations we presented a technique called "Universal Matrices Technique", or "Local Contribution Matrices Technique", where a mesh is constructed from geometric unitary tetrahedrons. In this article we show the MATHEMATICA Routine from which such unitary element matrices are worked out. It is our intention, as has already been done for the Helmholtz equation case, to work out a MATHEMATICA Notebook for a set of Einstein's field equations in the nearest future. Possible connections between our approach and others' such as Regge Calculus are outlined.

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Key words and phrases: local contribution matrices, partial differential equations, universal matrices.

## 1. Introduction – Universal Matrices Technique

In general relativity two main techniques have widely been used in order to treat Einstein's partial differential equations. They are named as Finite Difference and Finite Element Techniques. Finite Difference Techniques are based on approximation of derivatives in the differential equations.

One of the main problems faced by the users of FDT (Finite Difference Technique) is that the grid points are located along constant coordinate directions, which make this technique either not flexible enough or even inappropriate to lead with irregular physical domains. Finite Elements Technique (FET) is a method that approximates the solution of the continuum differential equation by a linear combination of trial functions. Through this method mesh refinements have frequently been used to conform to the boundaries and adapt to the variations in scales. Consequently, with FET is possible to use computational domains with irregular boundaries.

In the framework of Finite Elements, Silvester [8] introduced the formalism of the Analytical Integration Method as a technique throughout which by means of an appropriate coordinate transformation any finite element can be described as an unitary tetrahedron. A system of homogeneous coordinates is then used in order to simplify the integration process. In recent years, firstly Franco, Passaro and Machado [3] and then Franco, Passaro, Sircilli, Cardoso and Machado [4], have succeeded to apply the Silvester's analytical formalism to a coupled problem of differential partial diffusion and Helmholtz type equations. They presented computer techniques to facilitate calculations of local contribution matrices, and introduced a MATHEMATICA Notebook for computing tetrahedral finite element shape functions.

Following Franco's et al. work, two of us (Borges and Machado [2]) have just described how the technique of universal matrices might be applied to partial differential equations of Einstein type. In this brief communication we provide some details of how the trial linear equations in Einstein's equations are worked out and present a sketch for Einstein equations' MATHEMATICA Notebook (Schwarzschild space-time case).

## 2. Trial Linear Einstein Type Differential Equation and its MATHEMATICA Routine

In the case of a Schwarzschild space-time, one of the four remaining Einstein partial differential equations in the vacuum is given by

$$R_{11} = \gamma'' - \lambda'\gamma' - \gamma'^2 - \frac{2\lambda'}{r} = 0, \tag{1}$$

where $\gamma$ and $\lambda$ are functions of $r$ only, with primes denoting differentiations with respect to $r$. Equation (1) may be rewritten as ($\lambda = -u$):

$$\begin{aligned} R_{11} = {} & \frac{\partial^2 u}{\partial x^2}\left[1 + \frac{(y^2 + z^2)(x-1)}{x^3}\right] \\ & + \frac{\partial^2 u}{\partial y^2}\left[1 + \frac{(x^2 + z^2)(y-1)}{y^3}\right] \\ & + \frac{\partial^2 u}{\partial y^2}\left[1 + \frac{(x^2 + y^2)(z-1)}{z^3}\right] \\ & + (x^2 + y^2 + x^2) + \left\{\left(\frac{\partial u}{\partial x}\right)^2 \frac{1}{x^2} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{1}{y^2} + \left(\frac{\partial u}{\partial z}\right)^2 \frac{1}{z^2}\right. \\ & \left. + 4\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{1}{xy} + 4\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}\frac{1}{xz} + 4\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}\frac{1}{yz}\right\} \\ & + 2\left\{\frac{\partial u}{\partial x}\frac{1}{x} + \frac{\partial u}{\partial y}\frac{1}{y} + \frac{\partial u}{\partial z}\frac{1}{z}\right\} = 0. \end{aligned} \tag{2}$$

Equation $R_{11} = 0$ (2) is strongly non-linear. In order to use the finite elements technique we transform that equation in a set of trial linear equations. We shall then apply in (2) the weighted residual method, by defining:

$$R_{11}(\overline{u}) = r(x, y, z); \tag{3}$$

$$\overline{u} = \sum_i u_i a_i. \tag{4}$$

If a function $W$ is conveniently defined for the whole domain the weighted residual method imply that:

$$\int_{\Omega} r(x,\, y,\, z)\, w d\nu = 0. \tag{5}$$

Equation (2) may then be replaced by:

$$\begin{aligned} R_{11} = {} & \int_{\Omega} w\nabla^2\overline{u}\,d\nu + \int_{\Omega} w\frac{\partial^2\overline{u}}{\partial x^2}\left(\frac{(y^2+z^2)(x-1)}{x^3}\right)dV \\ & + \int_{\Omega} w\frac{\partial^2\overline{u}}{\partial y^2}\left(\frac{(x^2+z^2)(y-1)}{y^3}\right)dV + \int_{\Omega} w\frac{\partial^2\overline{u}}{\partial x^2}\left(\frac{(x^2+z^2)(z-1)}{z^3}\right)dV \\ & + \int_{\Omega} w\left(\frac{\partial\overline{u}}{\partial x}\right)^2\frac{x^2+y^2+z^2}{x^2}dV + \int_{\Omega} w\left(\frac{\partial\overline{u}}{\partial y}\right)^2\frac{x^2+y^2+z^2}{y^2}dV \\ & + \int_{\Omega} w\left(\frac{\partial\overline{u}}{\partial z}\right)^2\frac{x^2+y^2+z^2}{z^2}dV + \int_{\Omega} 4w\frac{\partial\overline{u}}{\partial x}\frac{\partial\overline{u}}{\partial y}\left(\frac{x^2+y^2+z^2}{xy}\right)dV \\ & + \int_{\Omega} 4w\frac{\partial\overline{u}}{\partial x}\frac{\partial\overline{u}}{\partial z}\left(\frac{x^2+y^2+z^2}{xz}\right)dV + \int_{\Omega} 4w\frac{\partial\overline{u}}{\partial x}\frac{\partial\overline{u}}{\partial y}\left(\frac{x^2+y^2+z^2}{yz}\right)dV \\ & + \int_{\Omega} 2w\frac{\partial\overline{u}}{\partial x}\frac{1}{x}dV + \int_{\Omega} 2w\frac{\partial\overline{u}}{\partial y}\frac{1}{y}dV + \int_{\Omega} 2w\frac{\partial\overline{u}}{\partial z}\frac{1}{z}dV = 0. \end{aligned} \tag{6}$$

The first term in the last equation may be developed by applying the Green identity, as follows:

$$\int_{\Omega} w\nabla^2\overline{u}\,d\nu = \oint_{\partial\Omega} w(\nabla\overline{u})\vec{n}\,dS - \int_{\Omega}(\nabla W)(\nabla\overline{u})\,dV. \tag{7}$$

The first term in (6), would then become:

$$\begin{aligned} \sum_i \int_{\Omega} wu_i \cdot \nabla^2\alpha_i dV &= \int_{\Omega} w\nabla^2\alpha_i dV \\ &= \sum_i u_i \oint_{\partial\Omega} w\nabla\alpha_i \cdot \vec{n}\,dS - \sum_i \int_{\Omega} u_i \cdot \nabla^2\alpha_i dV. \end{aligned} \tag{8}$$

If we impose $w = 0$ em $\partial\Omega$, we have

$$\sum_i u_i \int_\Omega w \nabla^2 \alpha_i dV = -\sum_i \int_\Omega \nabla w \cdot \nabla \alpha_i dV, \tag{9}$$

where $w = \sum_j w_j \alpha_j$, and consequently

$$\sum_i u_i \int_\Omega w \nabla^2 \alpha_i dV = -\sum_{i,\, j} w_i u_j \int_\Omega \nabla \alpha_i \cdot \nabla \alpha_j dV. \tag{10}$$

The first term in (6) is finally written as:

$$\int_\Omega w \nabla^2 \overline{u} dV = -\sum_{i,\, j} w_i u_j \int_\Omega \nabla \alpha_i \cdot \nabla \alpha_j dV. \tag{11}$$

For analogous procedure, the weighted residual method when applied to all terms in (6) produces sixteen trial linear functions, as follows:

$$R_{11} = \sum_i w_i \left\{ -\sum_j u_j \int_\Omega \nabla \alpha_i \nabla \alpha_j dxdydz \right.$$

$$- \sum_j u_j \int_\Omega \frac{(y^2 + z^2)(x - 1)}{x^3} \frac{\partial \alpha_i}{\partial x} \frac{\partial \alpha_j}{\partial x} dxdydz$$

$$- \sum_j u_j \int_\Omega \frac{(y^2 + z^2)(-2x^3 - 3x^2)}{x^6} \alpha_i \frac{\partial \alpha_j}{\partial x} dxdydz$$

$$- \sum_j u_j \int_\Omega \frac{(y^2 + z^2)(y - 1)}{y^3} \frac{\partial \alpha_i}{\partial y} \frac{\partial \alpha_j}{\partial y} dxdydz$$

$$- \sum_j u_j \int_\Omega \frac{(x^2 + z^2)(-2y^3 - 3y^2)}{y^6} \alpha_i \frac{\partial \alpha_j}{\partial y} dxdydz$$

$$- \sum_j u_j \int_\Omega \frac{(y^2 + y^2)(z - 1)}{z^3} \frac{\partial \alpha_i}{\partial z} \frac{\partial \alpha_j}{\partial z} dxdydz$$

$$- \sum_j u_j \int_\Omega \frac{(x^2 + y^2)(-2z^3 - 3z^2)}{z^6} \alpha_i \frac{\partial \alpha_j}{\partial z} dxdydz$$

$$
\begin{aligned}
&+ \sum_{j,\,k} u_j u_k \int_\Omega \alpha_i \frac{\partial \alpha_j}{\partial x} \frac{\partial \alpha_k}{\partial x} \frac{x^2 + y^2 + z^2}{x^2}\, dxdydz \\
&+ \sum_{j,\,k} u_j u_k \int_\Omega \alpha_i \frac{\partial \alpha_j}{\partial y} \frac{\partial \alpha_k}{\partial y} \frac{x^2 + y^2 + z^2}{y^2}\, dxdydz \\
&+ \sum_{j,\,k} u_j u_k \int_\Omega \alpha_i \frac{\partial \alpha_j}{\partial z} \frac{\partial \alpha_k}{\partial z} \frac{x^2 + y^2 + z^2}{z^2}\, dxdydz \\
&+ 4\sum_{j,\,k} u_j u_k \int_\Omega \alpha_i \frac{\partial \alpha_j}{\partial x} \frac{\partial \alpha_k}{\partial y} \frac{x^2 + y^2 + z^2}{xy}\, dxdydz \\
&+ 4\sum_{j,\,k} u_j u_k \int_\Omega \alpha_i \frac{\partial \alpha_j}{\partial x} \frac{\partial \alpha_k}{\partial z} \frac{x^2 + y^2 + z^2}{xz}\, dxdydz \\
&+ 4\sum_{j,\,k} u_j u_k \int_\Omega \alpha_i \frac{\partial \alpha_j}{\partial y} \frac{\partial \alpha_k}{\partial y} \frac{x^2 + y^2 + z^2}{yz}\, dxdydz \\
&+ 2\sum_{j} u_j \int_\Omega \alpha_i \frac{\partial \alpha_j}{\partial x} \frac{1}{x}\, dxdydz \\
&+ 2\sum_{j} u_j \int_\Omega \alpha_i \frac{\partial \alpha_j}{\partial y} \frac{1}{y}\, dxdydz \\
&+ 2\sum_{j} u_j \int_\Omega \alpha_i \frac{\partial \alpha_j}{\partial z} \frac{1}{z}\, dxdydz \Bigg\} = 0.
\end{aligned}
\tag{12}
$$

Through an appropriate coordinate transformation given by:

$$
\begin{aligned}
x &= x_1 = (x_2 - x_1)\xi + (x_3 - x_1)\vartheta + (x_4 - x_1)\zeta \\
y &= y_1 = (y_2 - y_1)\xi + (y_3 - y_1)\vartheta + (y_4 - y_1)\zeta \\
z &= z_1 = (z_2 - z_1)\xi + (z_3 - z_1)\vartheta + (z_4 - z_1)\zeta,
\end{aligned}
\tag{13}
$$

where $0 \le \xi \le 1$, $0 \le \vartheta \le 1$, and $0 \le \zeta \le 1$, and $(x_1, y_1, z_1) \ldots (x_4, y_4, z_4)$ are the four node cartesian coordinate of tetrahedron. All the sixteen

terms in (12) may be rewritten with the support of a MATHEMATICA 4.2 Software Package. The MATHEMATICA Routine is described below:

The MATHEMATICA Routine for

Schwarzschild Space-Time

(A sketch for a MATHEMATICA Notebook)

(Tetrahedral Finite Elements for the Schwarzschild equation $R_{11} = O$: a complete calculation of the matricial coefficients for the first approximation order)

K[A_B_C_]:= (((xl-Axl-Bxl-Cxl+Ax2+Bx3+Cx4)$^2$ +

(yl - Ayl - Byl - Cyl + Ay2 + By3 + CY4)$^2$) (-1 + zl - Azl - Bzl - Czl +Az2 + Bz3 + Cz4))/

(zl -Azl - Bzl - Czl + A z2 + Bz3 + Cz4)$^3$;

Integrate[K[A,B,C],{C,O,l-A-B}] 2 (l - A - B)

Integrate[%,{B,O,l-A}] (l - A)$^2$

Integrate[%,{A,0,1}1] 1 - 3 L[A_,B_,C_]:= (((xl - Axl - Bxl -Cxl + Ax2 + Bx3 + Cx4)$^2$ +

(yl.Ayl - Byl - Cyl + Ay2 + BY3 + CY4)$^2$) (3(zl - Azl - Bzl - Czl + Az2 + Bz3 + Cz4)$^2$ - .

2 (zl - Azl - Bzl - Czl + Az2 + Bz3 + Cz4)$^3$)/(zl - Azl - Bzl - Czl + Az2 + Bz3 + Cz4)$^6$;

Integrate[L[A,B,C],{C,O,l-A-B}] 2 (l -A - B) Integrate[%,{B,O, 1-A}] (1 - A)$^2$

Integrate[%{A,O,l}] 1 - 3 M[A_,B_,C_]:=

((xl - Axl - Bxl - Cxl + Ax2 + Bx3 + Cx4)$^2$ + (yl - Ayl - Byl - Cyl + Ay2 + By3 + Cy4)$^2$ +

(zl - Azl - Bzl - Czl + Az2 + Bz3 + Cz4)$^2$)/ (xl - Axl - Bxl - Cxl + Ax2 + Bx3 + Cx4)$^2$;

Integrate[M[A,B,C],{C,O,1-A-B}] 2 (1-A - B) Integrate[%,{B,O,1-A}] (l - A)$^2$

Integrate[%,{A,0,1}] 1- 3 P[A_,B_,C_]:=

((xl-Axl-Bxl-Cxl+Ax2+Bx3+Cx4)$^2$ + (yl - Ayl - Byl - Cyl + Ay2 + By3 + CY4)$^2$ +

(zl -Azl - Bzl - Czl + Az2 + Bz3 + Cz4)$^2$)/ (yl - Ayl - Byl - Cyl +Ay2 + By3 + Cy4)$^2$;

Integrate[P[A,B,C],{C,O,1-A-B}] 2 (1 - A - B) Integrate[%,{B,O,l-A}] (l -A)$^2$

```
Integrate[%,{A,O,I}] 1 - 3 Q[A_,B_,C_]:=

((xl - Axl - Bxl - Cxl + Ax2 + Bx3 + Cx4)^2 + (yl - Ayl - Byl - Cyl + Ay2 + By3 + CY4)^2 +

(zl - Azl - Bzl - Czl +Az2 + Bz3 + Cz4)^2)/(zl - Azl - Bzl - Czl + Az2 + Bz3 +Cz4)^2;

F[A_,B_,C_]:= ((xl - Axl - Bxl - Cxl + Ax2 + Bx3 + Cx4)^2 +

(yl - Ayl - Byl - Cyl + Ay2+ By3 + Cy4)^2 +(zl - Azl - Bzl- Czl +Az2 + Bz3 + Cz4)^2)/

((xl - Axl - Bxl - Cxl +A x2 + Bx3 +Cx4) (zl -Azl - Bzl -Czl +Az2 + Bz3 +Cz4));

Integrate[F[A,B,C],{C,O,1-A-B}] 2 (l -A - B) Integrate[%,{B,O, l-A}] (l - A)^2

Integrate[%,{A,0,1}] 1 - 3 G[A_,B_,C_]:=

((xl - Axl - Bxl - Cxl + Ax2 + Bx3 + Cx4)^2 +  (yl - Ayl - Byl - Cyl + Ay2 + By3 + CY4)^2 +

(zl -Azl - Bzl -Czl + Az2 + Bz3 + Cz4)^2) /

((yl - Ayl - BYl - CYl + Ay2 + BY3 + CY4)

(zl - Azl - Bzl - Czl + Az2 + Bz3 + Cz4));

Integrate[G[A,B,C],{C,O,l-A-B}] 2 (l -A - B) Integrate[%,{B,O,1-A}] (l - A)^2

Integrate[%,[A,O,1}] 1 - 3

W[A_,B_,C_]:= 1/(xl + A (-xl + x2) + B (-xl + x3) + C (-xl + x4));

Integrate[W[A,B,C],{C,O,l-A-B}]

Log[xl -Axl - Bxl +Ax2 + Bx3]              Log[Ax2 + Bx3 + x4 - Ax4 - x4]
.............................       -      ..............................
          xl - x4                                      xl - x4

Integrate[%,{B,O,1-A}]

(-xl + Axl - Ax2) Log[xl - Axl + Ax2]           x2 + x3 - Ax3) Log[Ax2+ x3 - Ax3]
.......................................    +    ..................................
        (-xl + x3) (x3 - x4)                         xl - xlx3 - xlx4 + x3x4

    (Ax2 +x4 -Ax4) Log[Ax2 +x4 -Ax4]
 -  ....................................
          (x3 - x4) (-xl + x4)
```

```
Integrate[%,{A,0,1}]

        2                                                   2
  xl Log[x1]                                          x2 Log[x2]
..........................................  +  ..........................................
2 (-xl + x2) (-xl + x3) (xl - x4)              2 (-xl + x2) (-x2 + x3) (-x2 + x4)

        2                                                   2
  x3 Log[x3]                                          x4 Log[x4]
+ ........................................  -  ..........................................
2 (xl - x3) (x2 - x3) (x3 - x4)                2 (xl - x4) (x2 - x4) (x3 - x4)
```

The first of the sixteen terms, in (12) is then rewritten as follows:

$$\int_{\Omega} \frac{(y^2+z^2)(x-1)}{x^3} \frac{\partial \alpha_i}{\partial x} \frac{\partial \alpha_j}{\partial x}\, dxdydz$$

$$= \frac{1}{36V^2} \int_0^1 d\xi \int_0^{1-\xi} d\vartheta \int_0^{1-\xi-\vartheta} A(\xi,\, \vartheta,\, \zeta) \sum_k \sum_m b_k b_m \frac{\partial \alpha_i}{\partial \vartheta_k} \frac{\partial \alpha_i}{\partial \vartheta_m}\, d\xi d\vartheta d\zeta, \quad (14)$$

where $A(\xi,\, \vartheta,\, \zeta)$ is given by

$$A(\xi,\, \vartheta,\, \zeta) = ((-1 + x_1 - \xi x_1 - \vartheta x_1 - \zeta x_1 + \xi x_2 + \vartheta x_3 + \zeta x_4)$$
$$((-1 + y_1 - \xi y_1 - \vartheta y_1 - \zeta y_1 + \xi y_2 + \vartheta y_3 + \zeta y_4)^2$$
$$+ (z_1 - \xi z_1 - \vartheta z_1 - \zeta z_1 + \xi z_2 + \vartheta z_3 + \zeta z_4)^2))/(x_1 - \xi x_1$$
$$- \vartheta x_1 - \zeta x_1 + \xi x_2 + \vartheta x_3 + \zeta x_4)^3 \quad (15)$$

and consequently:

$$\int_{\Omega} \frac{(y^2+z^2)(x-1)}{x^3} \frac{\partial \alpha_i}{\partial x} \frac{\partial \alpha_j}{\partial x}\, dxdydz = \frac{1}{36V^2} \frac{1}{3} b_k b_m, \quad (16)$$

where $1 \leq k \leq 4$ e $1 \leq m \leq 4$.

The last result is analogous to the first local contribution matrix obtained in an earlier paper by two of us (Borges and Machado [2]). All the remaining matrices may be obtained by using the MATHEMATICA routine presented here.

## 3. Concluding Remarks and Further Directions

Despite all the achievements brought by FDT and FET in the framework of partial differential equations, two other methods have been used for dealing with equations of Einstein type, the so-called Spectral Methods and Regge Calculus. Spectral Methods are used for studies of the gravitational collapse and for the construction of an initial data of a black hole collision (see, for instance, Bonazzola and Mark [1]). Unfortunately one of the disadvantages of Spectral Methods appears when applied to evolution equations; the Courant condition is, in this method, more restrictive than in the Finite Difference Technique. Regge Calculus, instead, represents an interesting method for numerical computation of Einstein equations. It may be regarded as a discrete version of Einstein's general relativity (Regge [5]; Willians and Tuckey [9] and Regge and Willians [6]). In this formalism all the dynamical variables are represented by finite distances; the space-time continuum is approximated by flat simplices with the curvature on the vertices. The applications of Regge Calculus, up to now, however, have not been able to show this technique as a competitive approach to treat with Einstein equations in terms of accuracy for a given computational effort.

Nevertheless through the use of the formalism of Universal Matrices such drawback of Regge Calculus might, al least, be attenuated or even disappear. All the simplices would now be unitary tetrahedrons, on which the homogeneous coordinates are used in order to simplify the integration process. The application of the Universal Matrices Technique to the Regge Calculus is under investigation by our research group and results will certainly be reported in future communications. We are also firmly intended to apply the results here reported to different problems concerned with Schwarzschild symmetries such as colliding black holes (Seidel [7]).

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# ON THE PROOF OF AKIMOVIC THEOREM

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( Received March 13, 2001 )

Submitted by Herb Silverman

## Abstract

In this note, we reprove the Akimovic theorem which plays an important role in the theory of Orlicz spaces.

The proof of Akimovic theorem [1] can be found in Wu et al. [7, pp. 29-31] or Chen [2, pp. 12-13]. In order to refine the proof, we need some definitions and lemmas.

**Definition 1.** An $\mathcal{N}$-function $M(u)$ is said to be *uniformly convex* for large $u$ (for small $u$) if, for each $a \in (0, 1)$, there are $\delta = \delta(a) > 0$ and $u_0 = u_0(a) > 0$ such that

$$M\left(\frac{u + bu}{2}\right) \leq \frac{1 - \delta}{2}[M(u) + M(bu)] \tag{1}$$

for $u \geq u_0$ $(0 \leq u \leq u_0)$ and $b \in [0, a]$. If (1) holds for all $u \geq 0$, we say that $M(u)$ is *uniformly convex* for all $u$.

**Lemma 1.** *Let* $M(u) = \int_0^{|u|} p(t)\, dt$ *be an $\mathcal{N}$-function. Then* $M(u)$ *is uniformly convex for large u (for small u) [for all u] iff for any given* $\varepsilon > 0$ *there are* $K = K(\varepsilon) > 1$ *and* $u_0 = u_0(\varepsilon) > 0$ *such that*

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$$p[(1+\varepsilon)u] \geq Kp(u) \tag{2}$$

*for* $u \geq u_0$ $(0 \leq u \leq u_0)$ $[u \geq 0]$.

The proof of Lemma 1 can be found in [6, pp. 284-286].

**Definition 2.** An $\mathcal{N}$-function $M(u)$ is said to be *strictly convex for all* $u$ (strictly convex, shortly) if for any $u, v \in \mathbb{R}$ with $u \neq v$

$$M\left(\frac{u+v}{2}\right) < \frac{1}{2}[M(u) + M(v)]. \tag{3}$$

It is known that $M(u)$ is strictly convex iff $M(u)$ is strictly convex on $[0, \infty)$, namely for any $u, v \geq 0$ with $u \neq v$ inequality (3) holds.

**Lemma 2** [5]. *Let* $M(u) = \int_0^{|u|} p(t)\,dt$ *and* $N(v) = \int_0^{|v|} q(s)\,ds$ *be a pair of complementary* $\mathcal{N}$*-functions. Then the following assertions are equivalent*:

(i) $M(u)$ *is strictly convex*;

(ii) $p(t)$ *is strictly increasing on* $[0, \infty)$;

(iii) $q(s)$ *is continuous on* $[0, \infty)$;

(iv) $N(v)$ *is a smooth function.*

**Definition 3.** An $\mathcal{N}$-function is said to satisfy $\Delta_2$*-condition for large* $u$ *(for small* $u$*) [for all* $u$*]*, in symbol $M \in \Delta_2(\infty)$ $(M \in \Delta_2(0))$ $[M \in \Delta_2]$, if there are $C > 2$ and $u_0 > 0$ such that $M(2u) \leq CM(u)$ for $u \geq u_0$ (for $0 \leq u \leq u_0$) [for all $u \geq 0$]. We write that $M \in \nabla_2(\infty)$ $(M \in \nabla_2(0))$ $[M \in \nabla_2]$ if its complementary $\mathcal{N}$-function $N \in \Delta_2(\infty)$ $(N \in \Delta_2(0))$ $[N \in \Delta_2]$.

**Lemma 3.** *For an* $\mathcal{N}$*-function* $M(u) = \int_0^{|u|} p(t)\,dt$*, define*

$$A_M = \liminf_{t\to\infty} \frac{tp(t)}{M(t)}, \qquad B_M = \limsup_{t\to\infty} \frac{tp(t)}{M(t)},$$

$$A_M^0 = \liminf_{t\to 0} \frac{tp(t)}{M(t)}, \qquad B_M^0 = \limsup_{t\to 0} \frac{tp(t)}{M(t)},$$

$$\overline{A}_M = \inf_{t>0} \frac{tp(t)}{M(t)}, \qquad \overline{B}_M = \sup_{t>0} \frac{tp(t)}{M(t)}. \tag{4}$$

*We have*

(i) $M \in \Delta_2(\infty) \cap \nabla_2(\infty)$ *iff* $1 < A_M \le B_M < \infty$;

(ii) $M \in \Delta_2(0) \cap \nabla_2(0)$ *iff* $1 < A_M^0 \le B_M^0 < \infty$;

(iii) $M \in \Delta_2 \cap \nabla_2$ *iff* $1 < \overline{A}_M \le \overline{B}_M < \infty$.

The proof of Lemma 3(i) is seen in [6, p. 26]. Now we can simplify the proof of the following Akimovic theorem.

**Theorem 1.** *Suppose that $\mathcal{N}$-function $M \in \Delta(\infty) \cap \nabla_2(\infty)$ $(M \in \Delta_2(0) \cap \nabla_2(0))$ $[M \in \Delta_2 \cap \nabla_2]$. Put*

$$M_0(u) = \int_0^{|u|} \frac{M(t)}{t}\,dt \tag{5}$$

*and let $N_0(v)$ be the complementary $\mathcal{N}$-function of $M_0(u)$. Then*

(i) *$M_0(u)$ and $M(u)$ are equivalent for all $u \ge 0$; $M_0(u)$ and $N_0(v)$ are strictly convex.*

(ii) *$M_0 \in \Delta_2(\infty) \cap \nabla_2(\infty)$ $(M_0 \in \Delta_2(0) \cap \nabla_2(0))$ $[M \in \Delta_2 \cap \nabla_2]$.*

(iii) *$M_0$ and $N_0$ are both uniformly convex for large arguments (for small arguments) [for all arguments].*

**Proof.** We only prove the theorem in the case when $M \in \Delta_2(\infty) \cap \nabla_2(\infty)$.

(i) By Lemma 6.2 in [5], $M_0 \sim M$ for $u \ge 0$. By (5), the right derivative of $M_0$, $p_0(u) = \dfrac{M(u)}{u}$ is continuous and strictly increasing on $[0, \infty)$ with $p_0(0) = \lim\limits_{u\to 0} \dfrac{M(u)}{u} = 0$. Hence, the right derivative $q_0(v)$ of

$N_0(v)$ is the inverse of $p_0(u)$, which is strictly increasing and continuous on $[0, \infty)$. In view of Lemma 2, $M_0$ and $N_0$ are strictly convex.

(ii) It follows from (i) and the assumption that $M_0 \in \Delta_2(\infty) \cap \nabla_2(\infty)$.

(iii) Let $a = \frac{1}{2}(1 + A_M)$ and $b = B_M + 1$. Then $1 < a < b < \infty$, by Lemma 3(i). Choose $u_0 > 0$ such that

$$a \le \frac{tp(t)}{M(t)} \le b \quad \text{for } t \ge u_0. \tag{6}$$

For any given $\varepsilon > 0$ and $u \ge u_0$, (6) implies that

$$\int_u^{(1+\varepsilon)u} \frac{a}{t}\,dt \le \int_u^{(1+\varepsilon)u} \frac{p(t)}{M(t)}\,dt \le \int_u^{(1+\varepsilon)u} \frac{b}{t}\,dt$$

or

$$\ln(1+\varepsilon)^a \le \ln\frac{M((1+\varepsilon)u)}{M(u)} \le \ln(1+\varepsilon)^b.$$

Therefore, we have

$$(1+\varepsilon)^a M(u) \le M((1+\varepsilon)u) \le (1+\varepsilon)^b M(u) \quad \text{for } u \ge u_0. \tag{7}$$

By the left side inequality of (7), one has for $u \ge u_0$

$$p_0((1+\varepsilon)u) = \frac{M((1+\varepsilon)u)}{(1+\varepsilon)u} \ge (1+\varepsilon)^{a-1}\,\frac{M(u)}{u} = (1+\varepsilon)^{a-1} p_0(u), \tag{8}$$

i.e., $M_0(u)$ is uniformly convex for large $u$ by Lemma 1. On the other hand, the right side inequality of (7) implies that for $u \ge u_0$

$$p_0((1+\varepsilon)u) = \frac{M((1+\varepsilon)u)}{(1+\varepsilon)u} \le (1+\varepsilon)^{b-1}\,\frac{M(u)}{u} = (1+\varepsilon)^{b-1} p_0(u). \tag{9}$$

Let $v_0 = p_0(u_0)$ and $v = p_0(u)$, then $u = q_0(v)$ for $v \ge v_0$ by (i). From (9) we have for $v \ge v_0$

$$(1+\varepsilon)q_0(v) \le q_0[(1+\varepsilon)^{b-1}v]. \tag{10}$$

For any $\varepsilon_1 > 0$, let $\varepsilon = (1+\varepsilon_1)^{\frac{1}{b-1}} - 1$. Then $\varepsilon > 0$ and $(1+\varepsilon)^{b-1} = 1+\varepsilon_1$. By (10), one has for $v \geq v_0$

$$q_0[(1+\varepsilon_1)v] \geq (1+\varepsilon_1)^{\frac{1}{b-1}} q_0(v).$$

Since $(1+\varepsilon_1)^{\frac{1}{b-1}} > 1$, $N_0(v)$ is uniformly convex for large $v$ by Lemma 1. The proof is complete.

**Remark 1.** Assume only that $M \in \nabla_2(\infty)$, then the left side inequalities of (6) as well as (7) are true, i.e., $M_0(u)$ is uniformly convex for large $u$. Similarly, if $M \in \Delta_2(\infty)$, then (10) holds and $N_0(v)$ is uniformly convex for large $v$.

**Lemma 4** (Kaminska [4]). *Let $L^{(M)}[0, 1]$, $L^{(M)}[0, \infty)$ and $l^{(M)}$ be the Orlicz function spaces and sequence space with Luxemburg norm. We have the following assertions*:

(i) $L^{(M)}[0, 1]$ *is uniformly rotund iff* (a) $M \in \Delta_2(\infty)$; (b) *$M$ is strictly convex and* (c) *$M$ is uniformly convex for large $u$.*

(ii) $L^{(M)}[0, \infty)$ *is uniformly rotund iff* (a) $M \in \Delta_2$; (b) *$M$ is strictly convex and* (c) *$M$ is uniformly convex for both large and small $u$.*

(iii) $l^{(M)}$ *is uniformly rotund iff* (a) $M \in \Delta_2(0)$; (b) *$M$ is strictly convex on $\left[0, M^{-1}\left(\frac{1}{2}\right)\right]$ and* (c) *$M$ is uniformly convex for small $u$.*

**Remark 2.** If $M(u)$ is strictly convex, then $M(u)$ is uniformly convex on any given closed interval $[u_1, u_2] \subset (0, \infty)$. The statement of Theorem 1.1 in Chen [2, p. 19] should be corrected in this way. Therefore, (b) and (c) in Lemma 4 (ii) are equivalent to that $M(u)$ is uniformly convex for all $u$. Conditions (b) and (c) in Lemma 4 (iii) are equivalent to that $M$ is uniformly convex on $\left[0, M^{-1}\left(\frac{1}{2}\right)\right]$.

From Theorem 1 and Lemma 4, we get another form of Akimovic theorem as follows:

**Theorem 2.** *Let* $X^{(M)} \in \{L^{(M)}[0, 1], L^{(M)}[0, \infty), l^{(M)}\}$. *If* $X^{(M)}$ *is* $L^{(M)}[0, \infty)$, *reflexive, then it admits an equivalent uniformly rotund norm, i.e,* $X^{(M)}$ *is super-reflexive.*

**Remark 3.** Enflo [3] proved that a Banach space $X$ is super-reflexive iff $X$ has an equivalent uniformly rotund norm. Theorem 2 is also true for those Orlicz spaces with Orlicz norm. Therefore, for the Orlicz spaces, super-reflexivity is equivalent to reflexivity. Here we omit the original definition and general properties of super-reflexive Banach spaces.

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# FORMULAS COUNTING THE TOTAL NUMBER OF ALL SELF-CONJUGATE PARTITIONS OF $n$

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Submitted by K. K. Azad

## Abstract

In this paper, we obtain new formulas counting the total number of all partitions of $n$ and self-conjugate partitions of $n$.

## 1. Introduction

We refer to [1, 5] for elementary definitions and notations. We define a self-conjugate partition of $S_n$. Kim and Park [2, 3] obtained the formula counting the total number of all partitions and self-conjugate partitions of $n$ and classified all self-conjugate partitions and described their forms. In this paper, we obtain new formulas counting the total number of all partitions of $n$ and self-conjugate partitions of $n$.

## 2. Definitions and Notations

In this section, we have some definitions and notations. The symmetric group on $\{1, 2, ..., n\}$ is denoted by $S_n$.

(1) $P[n]$ denotes the set of all partitions of $n$.

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(2) With every partition $[p] = [p_1, p_2, ..., p_n]$ in $P[n]$ we associate its conjugate partition $[p']$ [5, p. 129] obtained by interchanging rows and columns of the graph $[p]$; in other words, the terms of $[p']$ are given by the number of the nodes in columns of the graph $[p]$. (Evidently, the conjugate of $[p']$ is $[p]$.)

(3) A partition which is identical with its conjugate is said to be *self-conjugate* [5, p. 129]. For example, $4 + 2 + 1 + 1 = 8$ is self-conjugate,

$$(p_1 = 4,\ p_2 = 2,\ p_3 = p_4 = 1,\ p_5 = p_6 = p_7 = p_8 = 0).$$

(4) We define $P^*[n] = \{[p] \in P[n] : [p] \text{ is a self-conjugate partition}\}$.

## 3. Main Results

In this section, we obtain new formulas counting the total number of all partitions of $n$ and self-conjugate partitions of $n$.

**Notation.** $P(n, i)$ denotes the total number of all partitions of $n$ consisting of $i$ nodes in the first row.

**Example 1.** $P(10, 3) = 8$.

These partitions are as follows:

$$[3, 1^7], [3, 2, 1^5], [3, 2^2, 1^3], [3, 2^3, 1], [3^2, 1^4], [3^2, 2, 1^2], [3^2, 2^2], [3^3, 1].$$

**Proposition 1.** *Let* $n \geq 1$. *Then*

(1) $P(n, 1) = 1$, $P(n, n) = 1$.

(2) $P(n, m) = 0 (n < m)$.

**Proposition 2.** $|P[n]| = \sum_{i=1}^{n} P(n, i)$, *where* $|P[n]|$ *denotes the total number of all partitions of* $n$.

**Proposition 3.** $P(n, i) = \sum_{k=1}^{i} P(n - i, k)$.

**Theorem 4.** $P(n, i) = P(n - 1, i - 1) + P(n - i, i)$ $(n \geq 2, i \geq 2)$.

**Proof.** By Proposition 3,

$$P(n-1,\, i-1) = \sum_{k=1}^{i-1} P(n-i,\, k).$$

So

$$\begin{aligned} P(n,\, i) &= \sum_{k=1}^{i} P(n-i,\, k) \\ &= \sum_{k=1}^{i-1} P(n-i,\, k) + P(n-i,\, i) \\ &= P(n-1,\, i-1) + P(n-i,\, i). \end{aligned}$$

So from Theorem 4, we obtain the following result.

**Theorem 5.** $|P[n]| = \sum_{i=1}^{n} P(n, i) = \sum_{i=1}^{n} (P(n-1, i-1) + P(n-i, i))$.

**Example 2.** Let $a_{i,\,j} = P(i,\, j)$ and let $A = (a_{i,\,j})$ be an $n \times n$ matrix ($n = 15$). Then

$$A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 4 & 5 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 4 & 7 & 6 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 5 & 8 & 9 & 7 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 5 & 10 & 11 & 10 & 7 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 6 & 12 & 15 & 13 & 11 & 7 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\
1 & 6 & 14 & 18 & 18 & 14 & 11 & 7 & 5 & 3 & 2 & 1 & 1 & 0 & 0 \\
0 & 7 & 16 & 23 & 23 & 20 & 15 & 11 & 7 & 5 & 3 & 2 & 1 & 1 & 0 \\
1 & 7 & 19 & 27 & 30 & 26 & 21 & 15 & 11 & 7 & 5 & 3 & 2 & 1 & 1
\end{pmatrix}.$$

Since $a_{i,\,j} = a_{i-1,\,j-1} + a_{i-j,\,j}$ by Theorem 4, $P(11, 3) = a_{11,\,3} = a_{10,\,2} + a_{8,\,3}$

$= 5 + 5 = 10$. And since $|P[i]| = \sum_{k=1}^{i} P(i, k) = \sum_{k=1}^{i} a_{i,k}$ by Proposition 2, $|P[11]| = \sum_{k=1}^{11} a_{11,k} = 56$.

Thus, from the proofs of Theorems 2 and 4 of [2], we have the following new formulas.

**Theorem 6.** *Let $B(m)$ be the total number of all self-conjugate partitions of $n$ in which each partition has exactly $m$ columns and their columns have at least $m$ nodes. Then $B(m) = \sum_{k=1}^{m} P(a_m, k)$, where $a_m = \dfrac{n - m^2}{2}$.*

**Proof.** Since the total number of partitions of $n$ satisfying the conditions given in Theorem 6 is the total number of partitions of $\dfrac{n - m^2}{2}$ made in the first $m$ columns, $B(m) = \sum_{k=1}^{m} P(a_m, k)$, where $a_m = \dfrac{n - m^2}{2}$.

**Theorem 7.** *Let $P^*[n] = \{[p] : [p]$ is a self-conjugate partition$\}$ and let $B(m)$ be as in Theorem 6. Then*

$$|P^*[n]| = \sum_{\substack{1 \le m \le \sqrt{n} \\ m:\text{even}}} B_{a_m}(m), \qquad \textit{if } n \textit{ is even number},$$

$$|P^*[n]| = \left( \sum_{\substack{1 < m \le \sqrt{n} \\ m:\text{odd}}} B_{a_m}(m) \right) + 1, \quad \textit{if } n \textit{ is odd number}.$$

**Proof.** (1) Let $n$ be an even number. Then we cannot make the self-conjugate partitions in which each partition has exactly $2k+1$ columns and their columns have at least $2k+1$ nodes for $k = 0, 1, 2, \ldots$. If we make the self-conjugate partitions in which each partition has exactly $m$ columns and their columns have at least $m$ nodes, then the remaining nodes are $n - m^2$. Since the partitions which can be made with $\left[\dfrac{n - m^2}{2}\right]$

nodes in the first $i$ $(i \geq m+1)$ rows and the first $j$ $(j \leq m)$ columns are all self-conjugate partitions

$$B(m) = \sum_{k=1}^{m} P(a_m, k),$$

where $a_m = \dfrac{n - m^2}{2}$.

But $1 \leq m \leq \sqrt{n}$. Thus, the total number of all distinct self-conjugate partitions which can be made with $n$ nodes is given by

$$|P^*[n]| = \sum_{\substack{1 \leq m \leq \sqrt{n} \\ m:\text{even}}} B_{a_m}(m).$$

(2) Let $n$ be an odd number. Then we cannot make the self-conjugate partitions in which each partition has exactly $2k$ columns and their columns have at least $2k$ nodes for $k = 1, 2, \ldots$. So by the proof of (1), the total number of all distinct self-conjugate partitions which can be made with $n$ nodes is given by

$$|P^*[n]| = \left( \sum_{\substack{1 < m \leq \sqrt{n} \\ m:\text{odd}}} B_{a_m}(m) \right) + 1,$$

This completes the proof.

**Example 3.** Let $n = 31$. Then

$$\begin{aligned} |P^*[n]| &= 1 + B(3) + B(5) \\ &= 1 + \textstyle\sum_{k=1}^{3} P(11, k) + \sum_{k=1}^{5} P(3, k) \\ &= 1 + (1 + 5 + 10) + (1 + 1 + 1) \\ &= 20. \end{aligned}$$

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# A NOTE ON PRIME AND SEMIPRIME BI-IDEALS IN Γ-RINGS

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## Abstract

The purpose of this paper is to extend the results of prime and semiprime bi-ideals of rings to Γ-rings.

## 1. Introduction

In 1964, N. Nobusawa [9] defined a Γ-ring as follows:

Let $M$ and $\Gamma$ be two additive abelian groups. If for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, the conditions

(1) $x\alpha y \in M$ and $\alpha x \beta \in \Gamma$,

(2) $(x+y)\alpha z = x\alpha z + y\alpha z,\ x(\alpha+\beta)z = x\alpha z + x\beta z,\ x\alpha(y+z) = x\alpha y + x\alpha z$,

(3) $(x\alpha y)\beta z = x(\alpha y \beta)z = x\alpha(y\beta z)$,

(4) $x\alpha y = 0$ for all $x, y \in M$ implies that $\alpha = 0$,

are satisfied, then $M$ is called a Γ-*ring*.

Later Barnes [1] slightly weakened the defining conditions for Nobusawa's Γ-ring and defined Γ-ring as follows:

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Let $M$ and $\Gamma$ be two additive abelian groups. If for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, the conditions

(1) $x\alpha y \in M$,

(2) $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)z = x\alpha z + x\beta z$, $x\alpha(y+z) = x\alpha y + x\alpha z$,

(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

are satisfied, then $M$ is called a *$\Gamma$-ring*.

Usually, the former one is called *Nobusawa's $\Gamma$-ring* and the latter one is called *Barnes's $\Gamma$-ring*. The class of $\Gamma$-rings contains all $\Gamma$-rings. Coppage [2], Luh [7, 8], Kyuno [4, 5, 6] and Dutta [3] studied the structure of $\Gamma$-ring and obtained various generalizations of the corresponding parts in ring theory. In this paper we studied the prime and semiprime bi-ideals in Nobusawa's $\Gamma$-ring which generalize Roux's results. An additive subgroup $I$ of a $\Gamma$-ring $M$ is called a *right* (*left*) *ideal* of $M$ if $x\alpha y \in I$ $(y\alpha x \in I)$ for all $x$ in $I$, $\alpha$ in $\Gamma$ and $y$ in $M$. An additive group $I$ of a $\Gamma$-ring $M$ which is a left ideal as well as a right ideal is called an *ideal* of $M$. An ideal $P$ is called *prime* if for any ideals $A$ and $B$ of $M$,

$$A\Gamma B \subseteq P \Rightarrow A \subseteq P \text{ or } B \subseteq P.$$

An ideal $Q$ is called *semiprime* if for any ideal $I$ of $M$,

$$I\Gamma I \subseteq Q \Rightarrow I \subseteq Q.$$

## 2. Main Results

**Definition 1.** Let $M$ be a $\Gamma$-ring. A bi-ideal $B$ of $M$ is a *sub-$\Gamma$-ring* $B$ of $M$ satisfying $B\Gamma M\Gamma B \subseteq B$.

Every one-sided ideal of a $\Gamma$-ring is also a bi-ideal but not necessarily a one-sided ideal of $M$.

**Definition 2.** A bi-ideal $B$ of a $\Gamma$-ring $M$ is *prime* if $x\Gamma M\Gamma y \subseteq B$ implies $x \in B$ or $y \in B$.

**Definition 3.** A bi-ideal $B$ of a $\Gamma$-ring $M$ is *semiprime* if $x\Gamma M\Gamma x \subseteq B$ implies $x \in B$.

**Notations.** $x\Gamma M$ is the set of all finite sums $\sum x\alpha_i y_i$, and $M\Gamma y$ is the set of all finite sums $\sum x_i\alpha_i y$, where $x_i, y_i \in M$ and $\alpha_i \in \Gamma$.

**Theorem 4.** *A bi-ideal $B$ of a $\Gamma$-ring $M$ is prime if and only if $R\Gamma L \subseteq B$, with $R$ a right ideal of $M$, $L$ a left ideal of $M$, implies $R \subseteq B$ or $L \subseteq B$.*

**Proof.** Let $B$ be a prime bi-ideal of $M$ and let $R\Gamma L \subseteq B$. Suppose that $R \not\subseteq B$. For all $x \in L$ and $r \in R \setminus B$, we have $r\Gamma M\Gamma x \subseteq R\Gamma L \subseteq B$. Since $B$ is a prime and $r \notin B$, we have $x \in B$ for all $x \in L$, so $L \subseteq B$.

Conversely, suppose $R\Gamma L \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$ for any right ideal $R$ of $M$ and left ideal $L$ of $M$. Let $x, y \in M$ such that $x\Gamma M\Gamma y \subseteq B$. Then

$$(x\Gamma M)\Gamma(M\Gamma y) \subseteq x\Gamma M\Gamma y \subseteq B.$$

Since $x\Gamma M$ is a right ideal of $M$, and $M\Gamma y$ a left ideal of $M$, we have $x\Gamma M \subseteq B$ or $M\Gamma y \subseteq B$. Suppose $x\Gamma M \subseteq B$. Then $x\Gamma x \subseteq B$. Consider $(x)_r$ and $(x)_l$, the right ideal of $M$ and left ideal of $M$ generated by $x$ in $M$, respectively. Let $z$ be any element of the product $(x)_r\Gamma(x)_l$. Then

$$z = \sum_{i=1}^{n} \{m_i x + x\gamma a_i\}\,\delta\{k_i x + b_i\beta x\},$$

where $m_i, k_i \in Z$, $a_i, b_i \in M$, and $\alpha, \beta, \delta \in \Gamma$. Hence

$$z = \sum_{i=1}^{n} \{m_i k_i x\delta x + m_i x b_i\beta x + k_i x\gamma a_i\delta x + x\gamma a_i\delta b_i\beta x\}.$$

Since $x\Gamma x \subseteq B$ and $b_i\beta x$, $a_i\delta b_i\beta x$ are all the elements of $M$ and

$x\Gamma M \subseteq B$, we have that $z \in B$. Hence $(x)_r \Gamma (x)_l \subseteq B$. From our assumption it follows that $(x)_r \subseteq B$ or $(x)_l \subseteq B$ and hence $x \in B$. Similarly, if $M\Gamma y \subseteq B$, then $y \in B$. Hence $B$ is a prime ideal of $M$ and so the theorem is proved.

**Theorem 5.** *A prime bi-ideal of a $\Gamma$-ring $M$ is a prime one-sided ideal of $M$.*

**Proof.** Let $B$ be a prime bi-ideal of a $\Gamma$-ring $M$. It is only necessary to show that $B$ is a one-sided ideal of $M$. Clearly, by the definition,

$$(B\Gamma M)\Gamma(M\Gamma B) \subseteq B\Gamma M\Gamma B \subseteq B.$$

Since $B\Gamma M$ is a right ideal, and $M\Gamma B$ a left ideal of $M$, we have from Theorem 4, that $B\Gamma M \subseteq B$ or $M\Gamma B \subseteq B$. Hence $B$ is a one-sided ideal of $M$.

Let $B$ be any bi-ideal of a $\Gamma$-ring $M$ and let

$$L(B) = \{x \in B \mid M\Gamma x \subseteq B\}$$

and

$$H(B) = \{y \in L(B) \mid y\Gamma M \subseteq L(B)\}.$$

If $x \in L(B)$ and $z \in M$, then

$$z\Gamma x \subseteq M\Gamma x \subseteq B$$

and

$$M\Gamma z\Gamma x \subseteq M\Gamma(M\Gamma x) \subseteq M\Gamma x \subseteq B.$$

So, $L(B)$ is a left ideal of $M$ and $L(B) \subseteq B$.

**Theorem 6.** *If $B$ is any bi-ideal of a $\Gamma$-ring $M$, then $H(B)$ is the unique largest two-sided ideal of $M$ contained in $B$.*

**Proof.** Since $L(B) \subseteq B$ and $H(B) \subseteq L(B)$, we have that $H(B) \subseteq B$.

We now show that $H(B)$ is a two-sided ideal of $M$. Let $x \in H(B)$ and $y \in M$. Then $x \in B$ and since $x$ is also an element of $L(B)$, we have that $M\Gamma x \subseteq B$ and $x\Gamma M \subseteq L(B)$. Then $y\Gamma x \subseteq M\Gamma x \subseteq B$. So $y\Gamma x \subseteq B$. Furthermore $M\Gamma y\Gamma x \subseteq M\Gamma x \subseteq B$. So $y\Gamma x \subseteq L(B)$. Also $x\Gamma y \subseteq x\Gamma M \subseteq L(B)$. Hence $x\Gamma y \subseteq L(B)$. We must now prove that $y\Gamma x$ and $x\Gamma y \in H(B)$.

Now $x\Gamma y\Gamma M \subseteq x\Gamma M \subseteq L(B)$. Hence $x\Gamma y \subseteq H(B)$. Also

$$y\Gamma x\Gamma M \subseteq M\Gamma x\Gamma M \subseteq M\Gamma L(B) \subseteq L(B),$$

since $L(B)$ is a left ideal of $M$. Hence $y\Gamma x \subseteq H(B)$. Let $S$ be any ideal of $M$ and $S \subseteq B$, and let $u$ be an arbitrary element of $S$. Then $u \in B$ and $M\Gamma u \subseteq S \subseteq B$. Hence $S \subseteq L(B)$. Furthermore $u \in L(B)$ and $u\Gamma M \subseteq S \subseteq L(B)$. This implies that $u \in H(B)$ and hence $S \subseteq H(B)$.

**Theorem 7.** *Let $B$ be a prime bi-ideal of a $\Gamma$-ring $M$. Then $H(B)$ is a prime ideal of $M$.*

**Proof.** Let $B$ be a prime bi-ideal and let $X\Gamma Y \subseteq H(B)$ for any two-sided ideals $X$ and $Y$ of $M$. From Theorem 4 it follows that $X \subseteq B$ or $Y \subseteq B$ since $X\Gamma Y \subseteq B$. From Theorem 6 we have that $H(B)$ is the largest ideal in $B$. Hence $X \subseteq H(B)$ or $Y \subseteq H(B)$. It follows that $H(B)$ is a prime ideal of $M$.

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# ON THE $p$-INEXTENSIVITY OF BIVALENT TABLES WITH TWO COLUMNS

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**Abstract**

We recall the definition of $p$-extensive bivalent tables ($p$ natural number), then we investigate $p$-inextensive tables with two columns.

## 1. Introduction

This work is concerning bivalent tables which are defined by Fraisse [1] as follows.

**Definition 1.** A *bivalent table* is a system formed by two disjoint sets: a set $E$ of columns and a set $F$ of rows; and a function $T : E \times F \to \{+, -\}$ which, to each element in the Cartesian product $E \times F$, associates the value (+) or the value (−).

We usually say that $T$ is a *table* in $E \times F$.

**Definition 2.** Given two tables $T$ on $E \times F$ and $T'$ on $E' \times F'$, we say that $T$ is *embeddable* in $T'$, $T \leq T'$, if there exist an injection $e$ from $E$ into $E'$ and an injection $f$ from $F$ into $F'$, so that: $T'(e(x), f(y)) = T(x, y)$ for any $x$ in $E$ and any $y$ in $F$.

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**Definition 3.** The table $T$ is said to be *extensive* by the table $X$, relatively to rows, if either $T \le X$, or there exists an $X^+$ obtained from $X$ by adding a row, such that $T$ is not $\le X^+$. Otherwise, if $T$ is not $\le X$ and yet $T \le X^+$ for every $X^+$ obtained from $X$ by adding a row, then we say that $T$ is *inextensive* by $X$ relatively to rows.

**Definition 4.** A table $T$ is said to be *p-extensive* ($p$ natural number) if and only if $T$ is extensive by every table having $p$ rows. In other words, for every table $X$ having $p$ rows and such that $T$ is not $\le X$, there exists $X^+$ such that $T$ is not $\le X^+$.

**Definition 5.** $T$ is said to be *extensive relatively to rows* if $T$ is *extensive* by every table; it is said to be *inextensive* otherwise.

In the rest of the paper, we use *extensive* to mean *extensive relatively to rows*.

The essential aim of this work is the following problem that was proposed by Fraisse in 1986 [1] on the extension of bivalent tables.

*For each finite bivalent table, i.e., which has a finite set of rows and a finite set of columns, do there exist infinitely many integers p for which T is p-extensive; even does there exist an integer p such that T is q-extensive for every integer q greater than p.*

This problem of simple aspect is verified by a table $T$ having two columns only as we shall see below, which have not been investigated yet. We shall here treat some new cases in this work.

Fraisse gave in [1] the following example: a table $T$ having two columns and four rows, which is inextensive by the table $X$ having four columns and four rows;

$$T = \begin{array}{cc} + & + \\ - & - \\ + & - \\ + & - \end{array} \qquad X = \begin{array}{cccc} + & - & + & + \\ + & - & + & - \\ - & + & - & + \\ + & - & - & - \end{array}$$

In 1977, Lopez proved in [2] that: the above inextensive table $T$ is however $p$-extensive for every integer $p \geq 5$.

In the following three results that were obtained by Rauzy in [3], we denote $T(x, y, z, z')$ the table with two columns that contains $x$ rows $(+\,+)$, $y$ rows $(-\,-)$, $z$ rows $(+\,-)$ and $z'$ rows $(-\,+)$.

1. If $|z - z'| \leq 1$, then $T(x, y, z, z')$ is $p$-extensive for all $p$ equal or greater than the number of rows of $T$, $p \geq x + y + z + z'$.

2. The table $T(1, 1, n, 0)$ is $p$-extensive for $p \geq 6n - 4$.

This result match up the result of Lopez, that says that the table $T(1, 1, 2, 0)$ is $p$-extensive for $p \geq 5$. Here we obtain only $p \geq 8$, but we have any integer $n$ in the hypothesis instead of $n = 2$.

3. If $|z - z'| \geq 3$, then $T(x, y, z, z')$ is $p$-extensive for $p$ equal the number of rows of $T$; by against for $|z - z'| = 2$, the table $T$ is inextensive by at least a table of the same number of rows: and this generalizes the case of the table $T(1, 1, 2, 0)$ and (General criteria for inextensivity) the table given above.

## 2. General Inextensive Tables

In this work we shall give a theorem and some examples concerning inextensive bivalent tables of Fraisse [1]. We shall also verify the second and third results of Rauzy for tables $T(x, y, z, z')$ with $|z - z'| \geq 3$. Rauzy mentioned that this table is $p$-extensive for $p$ equal the number of rows of $T$. We improve this result by showing that the table $T(x, y, z, z')$ is $(p+1)$-extensive for $p$ equal the number of rows of $T$ and $T(x, y, z, z')$ is $(p+2)$-inextensive as we shall see next. We also provide some new examples about many cases and the general case for any table $T(x, y, z, z')$ such that $|z - z'| \geq 3$, and some $T(x, y, z, z')$ such that $|z - z'| = 2$.

We give this theorem for the general case of any bivalent table $T(1, 1, z, 0)$ which is $p$-inextensive.

**Theorem 1.** *The table* $T(1, 1, z, 0)$ *is* $(3z - 2)$*-inextensive, for* $z \geq 3$ *by the table U of the following form:*

$$U = \begin{array}{cccc} (+ & - & - & -)_{(z-1)} \\ (+ & + & - & -)_{(z-1)} \\ + & + & + & - \\ (- & - & + & -)_{(z-2)} \\ - & - & + & + \end{array}$$

*where* $(+\ -\ -\ -)_{(z-1)}$ *means that the table U contains* $(z-1)$ *times the row* $(+\ -\ -\ -)$.

**Proof.** We verify that the table $T$ is not embedding in $U$ (which depends on $z$). The first and second columns contain $(z-1)$ times the values $(+\,-)$; therefore $T$ is not $\leq U$ by these columns. Since the first and third columns do not contain the values $(-\,-)$, $T$ is not $\leq U$ by these columns. By the same method we can see that the first and fourth columns do not contain the values $(+\,+)$, therefore $T$ is not $\leq U$ by these columns.

For the second and third columns, we see that these columns contain $(z-1)$ times the values $(+\,-)$, and $(z-1)$ times the values $(-\,+)$, therefore $T$ is not $\leq U$ by these columns.

We see that the second and fourth columns do not contain the values $(+\,+)$, therefore $T$ is not $\leq U$ by these columns. At last, the third and fourth columns contain $(z-1)$ times the values $(+\,-)$, therefore $T$ is not $\leq U$ by these columns. Thus the table $T$ is not $\leq U$. To show that the table $T$ is $(3z-2)$-inextensive by the table $U$ for $z \geq 3$, we have to add all the possible rows $p$.

Firstly we cannot add the values $(+\,-)$ or the values $(-\,+)$ to the second and third columns, because if we add the values $(+\,-)$ to the

second and third columns, then the number of $(+\ -)$ will be equal to $z$, therefore the table $T \leq U^+$ by the second and third columns; contradiction. If we add the values $(-\ +)$ to the second and third columns, then the number of $(-\ +)$ will be equal to $z$, then the table $T \leq U^+$ by the second and third columns; contradiction. Therefore all the rows that contain the values $(+\ -)$ or the values $(-\ +)$ are not allowed rows because the table $T$ will be embedded in the table $U^+$.

Now it remains to add the rows which contain the values $(+\ +)$ or $(-\ -)$ to the second and third columns. Or $(+)$ to the fourth column, (because if not the number of $(+\ -)$ in the third and fourth columns will be equal to $z$, then $T \leq U^+$); and then the table $T$ will be embedded in the table $U^+$ by the second and fourth columns (because these columns lose the values $(+\ +)$); contradiction. Or we add the values $(-\ -)$ to the second and third columns, then we have to add the value $(-)$ to the first column, otherwise the number of $(+\ -)$ in the first and second columns will be equal to $z$, then $T \leq U^+$ by these columns; and then the table $T$ will be embedded in the table $U^+$ by the first and third columns.

We can give the following conjecture that gives the gap in the result of Rauzy [3] because Rauzy did mention that the table $T(1, 1, n, 0)$ is $p$-inextensive for all $p \geq 6n - 4$.

**Conjecture.** The table $T(1, 1, n, 0)$ is $p$-extensive for all tables with $p$ rows as soon as $p \geq 3n - 1$.

This conjecture will also generalize the result of Lopez [2].

We shall give some examples concerning the precedent theorem.

**Example 1.** The table $T(1, 1, 3, 0)$ is 7-inextensive by the table $X$ of four columns and seven rows as indicated below:

$$
X = \begin{array}{cccc}
+ & - & - & - \\
+ & - & - & - \\
+ & + & - & - \\
+ & + & - & - \\
+ & + & + & - \\
- & - & + & - \\
- & - & + & +
\end{array}
$$

**Proof.** Note that it is easy to check that the table $T(1, 1, 3, 0)$ is not embedded in $X$. We must add all the 16 rows to $X$ to see that $T$ will be embedded in $X^+$.

We cannot add different values $(+\,-)$ and $(-\,+)$ to the second and third columns respectively in $X$, without embedding $T$. Therefore, it is enough to add the values $(+\,+)$ and $(-\,-)$ to the second and third columns respectively. If we add the values $(+\,+)$ at these last columns, then we must add the value $(+)$ to the fourth column (because of the third and fourth columns); and then $T$ will be embedded in $X^+$ by the second and fourth columns; contradiction.

Now, if we add the values $(-\,-)$ to the second and third columns, then we must add the value $(-)$ to the first column (because of the first and second columns); and then $T$ is embedded in the table $X^+$ by the first and third columns; contradiction.

We can give this conjecture about the second result of Rauzy: $T(1, 1, n, 0)$ is $p$-extensive for $p \geq 6n - 4$.

**Conjecture.** The table $T(1, 1, 3, 0)$ is $p$-extensive for all $p \geq 8$.

Rauzy proved that the table $T(1, 1, 3, 0)$ is 12-extensive (relatively to rows) in relation to $p \geq 6n - 4$ (here $n = 3$).

**Example 2.** The table $T(1, 1, 4, 0)$ is 10-inextensive by the table $U$ of four columns and ten rows as indicated below:

$$U = \begin{matrix} + & - & - & - \\ + & - & - & - \\ + & - & - & - \\ + & + & - & - \\ + & + & - & - \\ + & + & - & - \\ + & + & + & - \\ - & - & + & - \\ - & - & + & - \\ - & - & + & + \end{matrix}$$

**Proof.** We must check that $T$ is not $\leq U$, we leave it to the reader.

We must show that any row added to $U$ gives a table $U^+$ in which $T$ is embedded. Now, we cannot add different values $(+\,-)$ and $(-\,+)$ to the second and third columns, because the table $T \leq U^+$ by these columns. Or we add the values $(+\,+)$ to the second and third columns, then we must add $(+)$ to the fourth column (because of the third and fourth columns); and then $T \leq U^+$ by the second and fourth columns; contradiction.

Or we add the values $(-\,-)$ to the second and third columns, then we must add $(-)$ to the first column (because of the first and second columns); and then $T \leq U^+$ by the first and third columns; contradiction.

We leave it to the reader to complete the proof.

We can give this conjecture about the second result of Rauzy.

**Conjecture.** The table $T(1, 1, 4, 0)$ is $p$-extensive for all $p \geq 11$.

Rauzy proved that the table $T(1, 1, 4, 0)$ is 20-extensive in relation to $p \geq 6n - 4$ (here $n = 4$).

## 3. New Inextensive Tables

We give in this section another type of table which is studied for the first time.

**Example 3.** The table $T(1, 2, 2, 0)$ is 5-*inextensive* by the table $X$ with five rows and five columns as indicated below:

$$X = \begin{matrix} + & - & - & - & - \\ + & + & - & - & - \\ + & + & + & - & - \\ - & - & + & + & + \\ - & - & - & + & - \end{matrix}$$

**Proof.** It is obvious to see that the table $T(1, 2, 2, 0)$ is not embeddable in table $X$. To show that the table $T$ is 5-inextensive by the table $X$, we must add all the possible 32 rows.

Firstly we are not able to add the values $(+\,-)$ and $(-\,+)$ to the second and third columns on $X$ without embedding $T$. Therefore, it is enough to add the values $(+\,+)$ and $(-\,-)$ to the second and third columns. If we add the values $(+\,+)$ at these second and third columns, then it is necessary to add the value $(+)$ to the fifth column (because of the third and fifth columns); and then $T$ is embedded in $X^+$ by the second and fifth columns; contradiction. On the other hand, if we add the values $(-\,-)$ to the second and third columns in $X$, then it is necessary to add $(-)$ to the first column (because of the first and second columns); and then $T$ is embedded in $X^+$ by the first and third columns; contradiction.

We give following conjecture about the table $T(1, 2, 2, 0)$.

**Conjecture.** The table $T(1, 2, 2, 0)$ is $p$-extensive for $p \geq 6$.

Now, we can generalize this result for any table $T(1, n, 2, 0)$ where $n$ is the number of rows $(- -)$ in $T$.

**Theorem 2.** *The table* $T(1, n, 2, 0)$ *is* $(3 + n)$*-inextensive by the table* $X$ *of the form*

$$X = \begin{array}{cccccc} + & - & - & - & - & (-)_{i+1} \\ + & + & - & - & - & \\ + & + & + & - & - & \\ - & - & + & + & + & \\ - & - & - & + & - & + \\ (-)_{i+2} & & & & & + \end{array}$$

*where* $3 \leq i \leq n,\ n \geq 3$.

**Proof.** Note that it is easy to check that $T(1, n, 2, 0)$ is not an embedding in $X$.

We see that in the first and second columns we have only one $(+ -)$. In the first and third columns we have only $(-)_{n+1}$ and $(+)_2$. Thus $T$ is not $\leq X$ (because $T$ has $(-)_{n+2}$ and one $(+)$ in the second column, so we lose one $(-)$).

In the first and fourth columns until the column number $(n + 1)$, we lose $(+ +)$. In the second and third columns, we have only one $(- +)$ and one $(+ -)$. In the second and the other columns, we lose $(+ +)$. We continue checking until the last two columns, $(n - 1)$ and $n$ columns, we see that, we have only one $(- +)$ and one $(+ -)$, so $T$ is not $\leq X$.

We want to prove that $T \leq X^+$ by adding all possible rows.

We shall prove it by mathematical induction for $n$.

Suppose that for every integer $n$, $P(n)$ is the proposition that involve $n$.

The proposition $P(3)$ is true as we shall see in the following.

To number the row from 0 to 63, every integer $< 64$ has to be developed in power of 2, then in binary numeration and then the code 0 is replaced by $(-)$ and the code 1 is replaced by $(+)$.

If we add one of the rows 0, 1, 2, 3, 4, 5, 6, 7, 16, 17, 18, 19, 20, 21, 22, 23, then $T \leq X^+$ by the first and third columns.

If we add one of the rows 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, then $T \leq X^+$ by the first and second columns.

If we add one of the rows 8, 9, 10, 11, 24, 25, 26, 27, 53, 54, 55, 56, 57, 58, then $T \leq X^+$ by the third and fourth columns.

If we add one of the rows 12, 13, 14, 15, 48, 49, 50, 51, then $T \leq X^+$ by the second and third columns.

If we add one of the rows 31, 59, 63, then $T \leq X^+$ by the fifth and sixth columns.

If we add one of the rows 28, 30, 52, 60, 62, then $T \leq X^+$ by the fourth and six columns.

If we add one of the rows 29, 61, then $T \leq X^+$ by the third and fifth columns.

Now, we must show that if $P(n)$ is true, then so is $P(n+1)$. First, we have the table $X$ of the form

$$
\begin{array}{lcccccccccc}
 & + & - & - & - & - & - & - & . & (-)_{n+1} & (-)_{n+2} \\
 & + & + & - & - & - & - & - & . & & \\
 & + & + & + & - & - & - & - & . & & \\
 & - & - & + & + & + & - & - & . & & \\
X = & - & - & - & + & - & + & - & . & & \\
 & - & - & - & - & - & + & + & . & & \\
 & - & - & - & - & - & - & + & . & & \\
 & . & . & . & . & . & . & . & . & + & - \\
(-)_{n+2} & & & & & & & & & + & + \\
(-)_{n+3} & & & & & & & & & - & +
\end{array}
$$

We know that for $n$, $T \le X^+$. Now we shall prove that it is also true for $(n+1)$.

Now, we cannot add the values $(+\,-)$ or the values $(-\,+)$ to the column number $n$ and number $(n+1)$ (because if not $T \le X^+$ by these columns, we lose $(-\,+)$ or $(+\,-)$); it remains to add the row which contains the values $(+\,+)$ or $(-\,-)$ to the $n^{\text{th}}$ column and the $(n+1)^{\text{th}}$ column. If we add the values $(+\,+)$ to these columns, or $(+)$ to the $(n-1)^{\text{th}}$ column (because if not $T \le X^+$ by the $(n-1)^{\text{th}}$ column and the $n^{\text{th}}$ column, we lose $(-\,+)$); and then $T \le X^+$ by the $(n-1)^{\text{th}}$ column and $(n+1)^{\text{th}}$ column (we lose $(+\,+)$); contradiction.

Finally, if we add $(-\,-)$ to the $n^{\text{th}}$ column and $(n+1)^{\text{th}}$ column, or $(-)$ to the $(n-1)^{\text{th}}$ column (because if not $T \le X^+$ by $(n-1)^{\text{th}}$ and $n^{\text{th}}$ columns, we lose $(+\,-)$); we continue adding $(-)$ to every column until we arrive from fourth to first column, we shall study these cases as we see in the above table.

We cannot add the values $(--)$ to the first and third columns (because if not $T \leq X^+$ by these columns); we have two choices $(+-)$ or $(-+)$. If we add $(+-)$ to the first and third columns, we put $(+)$ to the second column (because if not $T \leq X^+$ by the first and second columns, we lose $(+-)$); and then $T \leq X^+$ by the second and third columns (we lose $(-+)$). If we add $(-+)$ to the first and third columns, we put $(+)$ to the second column (because if not $T \leq X^+$ by the second and third columns, we lose $(-+)$); and then $T \leq X^+$ by third and fourth columns (we lose $(+-)$). Thus, $P(n+1)$ is true, and so the theorem is proved.

**4.** We give here another type of table $T(x, y, z, z')$ that contains $|z - z'| = 2$, which is inextensive by at least a table of the same number of rows of $T$.

**Example 4.** The table $T(1, 1, 3, 1)$ is 6-*inextensive* by the following table $U$ with six rows and five columns:

$$U = \begin{matrix} + & - & + & - & - \\ - & + & - & - & + \\ - & + & + & - & + \\ + & - & - & + & - \\ + & + & - & - & + \\ + & - & + & + & - \end{matrix}$$

**Proof.** Firstly, we verify that $T$ is not embedded in $U$. (We may leave it to reader for checking.)

To show that the table $T$ is inextensive by the table $U$ (relatively to row) we must add to $U$ all the 32 possible rows.

To number the rows from 0 to 31, every integer $< 32$ is developed in power of 2, then in binary numeration and then the code 0 is replaced by $(-)$ and the code 1 is replaced by $(+)$. For example, the number

$10 = 8 + 2 = 2^3 + 2^1$ is noted 01010 in binary numeration, then it is translated by the row $(- + - + -)$.

If we add one of the rows 0, 1, 2, 3, 4, 5, 6, then $T$ is embedded in $U^+$ by the first and second columns.

If we add one of the rows 7, 8, 9, 10, 11, 20, 21, 22, 23, 24, 25, 26, 27, then $T$ is embedded in $U^+$ by the second and third columns.

If we add one of the rows 15, 19, 31, then $T$ is embedded in $U^+$ by the fourth and fifth columns.

If we add one of the rows, 16, 17, 18, then $T$ is embedded in $U^+$ by the first and third columns.

If we add one of the rows, 12, 13, 28, 29, then $T$ is embedded in $U^+$ by the third and fourth columns.

If we add one of the rows, 14, 30, then $T$ is embedded in $U^+$ by the third and fifth columns.

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# REPRESENTATIONS OF $\mathcal{O}_n$ AND THEIR RESTRICTIONS TO $\mathrm{UHF}_n$

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## Abstract

We study a state $\rho$ of $\mathcal{O}_n$ satisfying some conditions and its restriction $\overline{\rho}$ to $\mathrm{UHF}_n$. For a representation $\pi_{\overline{\rho}}$ of $\mathrm{UHF}_n$ and a representation $\pi_\rho$ of $\mathcal{O}_n$ given by the GNS construction, we show that representations $\pi_{\overline{\rho}}$ and $\overline{\pi_\rho}$ of $\mathrm{UHF}_n$ are unitary equivalent, where $\overline{\pi_\rho}$ is the restriciton of $\pi_\rho$ to $\mathrm{UHF}_n$. Using this, we prove that the Cuntz state is pure. We also study the equivalent conditions for $\pi_1(\mathcal{O}_n) = \pi_2(\mathcal{O}_n)$ and those for $\pi_1 \mid \mathrm{UHF}_n = \pi_2 \mid \mathrm{UHF}_n$, where $\pi_1$ and $\pi_2$ are non-degenerate representations of $\mathcal{O}_n$. As an application, we obtain a one-to-one correspondence between the set of endomorphisms of $\mathcal{O}_n$ and that of unitaries in $\mathcal{O}_n$.

## 1. Introduction

J. Cuntz [4] introduced the *Cuntz algebra* $\mathcal{O}_n$ which is the

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$C^*$-algebra generated by $n = 2, 3, \ldots$ isometries $s_1, s_2, \ldots, s_n$ satisfying the Cuntz relations:

$$s_i^* s_j = \delta_{ij} I \quad \text{and} \quad \sum s_i s_i^* = I. \tag{1.1}$$

The Cuntz algebra $\mathcal{O}_n$ is a simple infinite $C^*$-algebra and it depends only on $n$ and not on the choice of the generators. A $\mathrm{UHF}_n$ *algebra* is a uniformly hyperfinite algebra $\bigotimes_{i=1}^{\infty} M_n$, where $M_n$ is the $n \times n$ matrix algebra (see [5]), and we consider it as a subalgebra of $\mathcal{O}_n$.

For the study of the Cuntz algebra $\mathcal{O}_n$, the representation theory has been developed (see [1], [2], [3], [7]). It is known that there is a correspondence between representations of $\mathcal{O}_n$ and endomorphisms of $\mathcal{B}(\mathcal{H})$ of Powers index $n$ up to unitary action, where $\mathcal{B}(\mathcal{H})$ is the $C^*$-algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. If a non-degenerate representation $\pi$ of $\mathcal{O}_n$ is irreducible, then the corresponding endomorphism is ergodic. Moreover, if the restriction $\pi|_{\mathrm{UHF}_n}$ of $\pi$ is irreducible, then the corresponding endomorphism is a shift of Powers index $n$ (see [3]).

In fact, the representation theory of a $C^*$-algebra is closely related to states of it. Since pure states of a $C^*$-algebra give irreducible representations by the GNS construction, it is natural that the study of pure states of $\mathcal{O}_n$ in connection with representations of $\mathcal{O}_n$ is one of our concerns.

Recall that a *state* of a $C^*$-algebra is a positive linear functional of norm one. Here, we consider a linear functional on $\mathcal{O}_n$ associated to a unit vector $\eta = (\eta^1, \eta^2, \ldots, \eta^n)$ in $\mathbb{C}^n$. Note that the formula $\eta^1\overline{\eta^1} + \cdots + \eta^n\overline{\eta^n} = 1$ and the Cuntz relation $\sum s_i s_i^* = I$ are naturally related. So, we define the *linear functional* $\omega$ *on* $\mathcal{O}_n$ *associated to a unit vector* $\eta = (\eta^1, \eta^2, \ldots, \eta^n) \in \mathbb{C}^n$ by

$$\omega(s_{i_1} \cdots s_{i_k} s_{j_l}^* \cdots s_{j_1}^*) = \eta^{i_1} \ldots \eta^{i_k} \overline{\eta^{j_l}} \ldots \overline{\eta^{j_1}}.$$

Let us point out that this linear functional $\omega$ is called the *Cuntz State* in [3] and they assert that it is pure. The proof that $\omega$ is a pure state may be known to specialists. The authors, however, cannot find a literature that contains it.

This paper begins with the investigation of the state $\rho$ of $\mathcal{O}_n$ with some conditions which come from the linear functional associated to a unit vector and its restriction $\overline{\rho}$ to $\mathrm{UHF}_n$. We have a representation $\pi_\rho$ of $\mathcal{O}_n$ on the Hilbert space $\mathcal{H}_\rho$ and a representation $\pi_{\overline{\rho}}$ of $\mathrm{UHF}_n$ by the GNS construction. Let $\overline{\pi_\rho}$ be the restriction of $\pi_\rho$ to $\mathrm{UHF}_n$. We show that two representations $\pi_{\overline{\rho}}$ and $\overline{\pi_\rho}$ of $\mathrm{UHF}_n$ are unitary equivalent. As an application, we prove that the Cuntz state is pure.

On the other hand, we study the characterization of representations of $\mathcal{O}_n$ and their restrictions to $\mathrm{UHF}_n$. For two non-degenerate representations $\pi_1$, $\pi_2$ of $\mathcal{O}_n$ on $\mathcal{H}$, there exist associated unitary operators $U \in \mathcal{B}(\mathcal{H})$ and $M \in M_n(\mathcal{B}(\mathcal{H}))$, where $M_n(\mathcal{B}(\mathcal{H}))$ is the set of all $n \times n$ matrices over $\mathcal{B}(\mathcal{H})$. We investigate the equivalent conditions for $\pi_2(\mathcal{O}_n) \subset \pi_1(\mathcal{O}_n)$ in terms of these unitary operators. Furthermore, we find the equivalent conditions for $\pi_1(\mathcal{O}_n) = \pi_2(\mathcal{O}_n)$. We also find the equivalent conditions for $\pi_1|_{\mathrm{UHF}_n} = \pi_2|_{\mathrm{UHF}_n}$. Finally, we apply our result to endomorphisms of $\mathcal{O}_n$ and give a one-to-one correspondence between the set of endomorphisms of $\mathcal{O}_n$ and that of unitaries in $\mathcal{O}_n$.

## 2. Representations of $\mathrm{UHF}_n$ Related to a State of $\mathcal{O}_n$

This section contains the study of two representations of $\mathrm{UHF}_n$ which are unitary equivalent and its application to the proof that the Cuntz state is pure.

We examine a simple infinite $C^*$-algebra generated by isometries

satisfying the Cuntz relations (1.1). In fact, the Cuntz algebra $\mathcal{O}_n$ is the $C^*$-algebra obtained as the closure of the linear span of operators $s_{i_1} \cdots s_{i_k} s_{j_l}^* \cdots s_{j_1}^*$. A subalgebra $\mathrm{UHF}_n$ of the Cuntz algebra $\mathcal{O}_n$ is the closure of the linear span of operators $s_{i_1} s_{i_2} \cdots s_{i_k} s_{j_{k-1}}^* \cdots s_{j_1}^*$. For a monomial $x = s_{i_1} s_{i_2} \cdots s_{i_k}$ in $\mathcal{O}_n$, the length $|x|$ is $k$ and $S_k$ denotes the set of all monomials of length $k$.

From the Cuntz relations (1.1), we can easily get the following lemma.

**Lemma 2.1.** *For any* $k \in \mathbb{N}$, *we have* $\sum_{x \in S_k} xx^* = I$.

**Proof.** Since $\sum s_i s_i^* = I$, we have $\sum_{x \in S_1} xx^* = I$ and

$$\sum_{x \in S_2} xx^* = \sum_{i,\, j=1}^{n} s_i s_j s_j^* s_i^* = \sum_{i=1}^{n} s_i \left( \sum_{j=1}^{n} s_j s_j^* \right) s_i^* = \sum_{i=1}^{n} s_i s_i^* = I.$$

When $k \geq 3$, the proof is similar to the case $k = 2$.

Here our concern is a linear functional on $\mathcal{O}_n$ satisfying some conditions which come from the linear functional associated to a unit vector in $\mathbb{C}^n$.

We first consider the canonical endomorphism $\psi$ of $\mathcal{O}_n$ defined by

$$\psi(z) = \sum_{i=1}^{n} s_i z s_i^*, \quad z \in \mathcal{O}_n$$

and notice that $\psi|_{\mathrm{UHF}_n}$ is one-sided shift (see [3]). We now let $\rho$ be a linear functional on $\mathcal{O}_n$ satisfying $\rho(I) = 1$ and

$$\rho(xy) = \rho(x)\rho(\psi^{|x|}(y)), \quad \rho(xy^*) = \rho(x)\overline{\rho(y)} \tag{2.1}$$

for any monomials $x$, $y$ in $\mathcal{O}_n$.

It is easily checked by routine arguments that the linear functional $\omega$ on $\mathcal{O}_n$ associated to a unit vector in $\mathbb{C}^n$ satisfies (2.1).

In the following two lemmas, $\rho$ is a linear functional on $\mathcal{O}_n$ satisfying $\rho(I) = 1$ and (2.1).

**Lemma 2.2.** *For any* $k \in \mathbb{N}$, *we have* $\sum_{x \in S_k} |\rho(x)|^2 = 1$.

**Proof.** Since $\sum_{x \in S_k} xx^* = I$, by Lemma 2.1, we have

$$\sum_{x \in S_k} |\rho(x)|^2 = \sum_{x \in S_k} \rho(xx^*) = \rho\left(\sum_{x \in S_k} xx^*\right) = \rho(I) = 1.$$

**Lemma 2.3.** *For any* $k, l \in \mathbb{N}$, *there exists* $y \in S_l$ *with* $\rho(\psi^k(y)) \neq 0$.

**Proof.** It follows from (2.1) that $|\rho(xy)|^2 = |\rho(x)|^2 |\rho(\psi^{|x|}(y))|^2$.

For any $k, l \in \mathbb{N}$, since $\sum_{\substack{x \in S_k \\ y \in S_l}} (xy)(xy)^* = \sum_{x \in S_k} x\left(\sum_{y \in S_l} yy^*\right)x = I$, by Lemma 2.2, we have

$$1 = \sum_{\substack{x \in S_k \\ y \in S_l}} |\rho(xy)|^2 = \sum_{\substack{x \in S_k \\ y \in S_l}} |\rho(x)|^2 |\rho(\psi^{|x|}(y))|^2$$

$$= \sum_{x \in S_k} |\rho(x)|^2 \sum_{y \in S_l} |\rho(\psi^k(y))|^2 = \sum_{y \in S_l} |\rho(\psi^k(y))|^2.$$

Thus we conclude that there exists $y \in S_l$ with $\rho(\psi^k(y)) \neq 0$.

From now on, we restrict our attention to the case that $\rho$ is a state of $\mathcal{O}_n$ satisfying (2.1). Since $\rho$ is positive, we get that $\rho(z^*) = \overline{\rho(z)}$ and $\rho(\psi^l(z^*)) = \overline{\rho(\psi^l(z))}$ for any $z \in \mathcal{O}_n$.

Let $(\pi_\rho, \mathcal{H}_\rho, 1 + N_\rho)$ be the cyclic representation of $\mathcal{O}_n$ induced by $\rho$ and $\overline{\pi_\rho}$ the restriction $\pi_\rho|_{\mathrm{UHF}_n}$ of $\pi_\rho$ to $\mathrm{UHF}_n$. We recall that for the left ideal $N_\rho = \{x \in \mathcal{O}_n \mid \rho(x^*x) = 0\}$, $\mathcal{H}_\rho$ is the Hilbert space obtained

as the completion of the pre-Hilbert space $\mathcal{O}_n / N_\rho$ and $1 + N_\rho \in \mathcal{H}_\rho$ is a cyclic vector for $\pi_\rho$.

The following lemma is very useful to prove the main theorem.

**Lemma 2.4.** *With the notations as above,* $1 + N_\rho$ *is a cyclic vector for* $\overline{\pi_\rho}$.

**Proof.** It is enough to show that for any $x = s_{i_1}s_{i_2}\cdots s_{i_l}$, $y = s_{j_1}s_{j_2}\cdots s_{j_k}$ in $\mathcal{O}_n$, there exist $\lambda \in \mathbb{C}$ and $y_1 \in \mathcal{O}_n$ with $|y_1| = l$ satisfying $\lambda x y_1^* - xy^* \in N_\rho$.

To do this, when $|x| = |y|$, we take $y_1 = y$ and $\lambda = 1$.

When $|x| > |y|$. By Lemma 2.3, there exists $y_2 \in \mathcal{O}_n$ with $\rho(\psi^l(y_2)) \neq 0$ and $|y_2| = l - k$. If we take $y_1 = yy_2$ and $\lambda = \overline{\rho(\psi^k(y_2))}^{-1}$, then the equality

$$(\lambda x y_1^* - xy^*)^*(\lambda x y_1^* - xy^*) = |\lambda|^2 yy_2y_2^*y^* - \overline{\lambda} yy_2y^* - \lambda yy_2^*y^* + yy^*$$

gives

$$\begin{aligned}
&\rho((\lambda x y_1^* - xy^*)^*(\lambda x y_1^* - xy^*)) \\
&= \rho(|\lambda|^2 yy_2y_2^*y^*) - \rho(\overline{\lambda} yy_2y^*) - \rho(\lambda yy_2^*y^*) + \rho(yy^*) \\
&= \rho(yy^*)\{|\lambda|^2|\rho(\psi^k(y_2))|^2 - \overline{\lambda}\rho(\psi^k(y_2)) - \lambda\rho(\psi^k(y_2^*)) + 1\} \\
&= \rho(yy^*)|\overline{\lambda}\rho(\psi^k(y_2)) - 1|^2 = 0
\end{aligned}$$

which implies $\lambda x y_1^* - xy^* \in N_\rho$.

When $|x| < |y|$. If we let $y_1 = s_{j_1}s_{j_2}\cdots s_{j_l}$, $y_2 = s_{j_{l+1}}\cdots s_{j_k}$ and $\lambda = \overline{\rho(\psi^l(y_2))}$, then from the equality

$$(\lambda x y_1^* - xy^*)^*(\lambda x y_1^* - xy^*) = |\lambda|^2 y_1y_1^* - \overline{\lambda} y_1y_2^*y_1^* - \lambda y_1y_2y_1^* + y_1y_2y_2^*y_1^*,$$

we have

$$\rho((\lambda x y_1^* - x y^*)^*(\lambda x y_1^* - x y^*))$$

$$= \rho(y_1 y_1^*)\{|\lambda|^2 - \overline{\lambda}\rho(\psi^l(y_2^*)) - \lambda\rho(\psi^l(y_2)) + \rho(\psi^l(y_2 y_2^*))\}$$

$$= \rho(y_1 y_1^*)|\overline{\lambda} - \rho(\psi^l(y_2))|^2 = 0$$

which implies $\lambda x y_1^* - x y^* \in N_\rho$.

Now, we consider the restriction $\rho|_{\mathrm{UHF}_n}$ of a state $\rho$ of $\mathcal{O}_n$ and denote it by $\overline{\rho}$. Since $\overline{\rho}$ is a state of $\mathrm{UHF}_n$, we have a representation $\pi_{\overline{\rho}}$ of $\mathrm{UHF}_n$ on $\mathcal{H}_{\overline{\rho}}$ with a cyclic vector $1 + N_{\overline{\rho}} \in \mathcal{H}_{\overline{\rho}}$ for $\pi_{\overline{\rho}}$. We are now in a position to state and prove the main theorem.

**Theorem 2.5.** *With the notations as above, two representations* $\overline{\pi_\rho}$ *and* $\pi_{\overline{\rho}}$ *of* $\mathrm{UHF}_n$ *are unitary equivalent.*

**Proof.** For any $x \in \mathrm{UHF}_n$, since we have

$$\rho(x) = \langle \pi_\rho(x)(1 + N_\rho), 1 + N_\rho \rangle = \langle \overline{\pi_\rho}(x)(1 + N_\rho), 1 + N_\rho \rangle$$

and $\overline{\rho}(x) = \langle \pi_{\overline{\rho}}(x)(1 + N_{\overline{\rho}}), 1 + N_{\overline{\rho}} \rangle$, it follows that

$$\langle \overline{\pi_\rho}(x)(1 + N_\rho), 1 + N_\rho \rangle = \langle \pi_{\overline{\rho}}(x)(1 + N_{\overline{\rho}}), 1 + N_{\overline{\rho}} \rangle.$$

From Lemma 2.4, $1 + N_\rho$ is a cyclic vector for $\overline{\pi_\rho}$ and $1 + N_{\overline{\rho}}$ is a cyclic vector for $\pi_{\overline{\rho}}$. Hence we conclude that $\overline{\pi_\rho}$ and $\pi_{\overline{\rho}}$ are unitary equivalent.

In the following, recall that the commutant $S'$ of a subset $S$ of $\mathcal{B}(\mathcal{H})$ is the set $\{y \in \mathcal{B}(\mathcal{H}) \mid xy = yx, \forall x \in S\}$ and $S'' = (S')'$.

**Corollary 2.6.** *Let* $\rho$ *be a state of* $\mathcal{O}_n$ *satisfying* (2.1). *If the restriction* $\overline{\rho}$ *of* $\rho$ *to* $\mathrm{UHF}_n$ *is pure, then we have* $\overline{\pi_\rho}(\mathrm{UHF}_n)'' = \mathcal{B}(\mathcal{H}_\rho)$. *In particular,* $\rho$ *is also pure.*

**Proof.** If $\overline{\rho}$ is a pure state of $\mathrm{UHF}_n$, then the representation $\pi_{\overline{\rho}}$ of $\mathrm{UHF}_n$ is irreducible. By Theorem 2.5, $\overline{\pi_\rho}$ is also an irreducible representation of $\mathrm{UHF}_n$ on $\mathcal{H}_\rho$ which implies that $\overline{\pi_\rho}(\mathrm{UHF}_n)'' = \mathcal{B}(\mathcal{H}_\rho)$.

Since $\overline{\pi_\rho}(\mathrm{UHF}_n)''$ is a subalgebra of $\pi_\rho(\mathcal{O}_n)''$, we have $\pi_\rho(\mathcal{O}_n)'' = \mathcal{B}(\mathcal{H}_\rho)$. Thus $\pi_\rho$ is irreducible and equivalently $\rho$ is pure.

Now, we give a proof that if the linear functional on $\mathcal{O}_n$ associated to a unit vector is a state, then it is a pure state.

**Corollary 2.7.** *Let $\omega$ be the linear functional on $\mathcal{O}_n$ associated to a unit vector. If $\omega$ is a state, then it is pure.*

**Proof.** Let $\omega$ be the linear functional on $\mathcal{O}_n$ associated to a unit vector $\eta$ in $\mathbb{C}^n$. Then $\omega$ satisfies (2.1).

Suppose that $\omega$ is a state of $\mathcal{O}_n$. It is straightforward to see that the restriction $\overline{\omega}$ of $\omega$ to $\mathrm{UHF}_n$ is the infinite tensor product state $\omega_\eta \otimes \omega_\eta \otimes \cdots$ of a vector state $\omega_\eta$ of a matrix algebra $M_n$ defined by $\omega_\eta(\cdot) = \langle \cdot\, \eta, \eta \rangle$ (see [8]). Since a vector state of $M_n$ is pure and the tensor product of vector states is also pure (see [6]), the restriction $\overline{\omega}$ is pure. Thus by Corollary 2.6, $\omega$ is also pure.

We finally point out that since the Cuntz state in [3] is a linear functional on $\mathcal{O}_n$ associated to a unit vector in $\mathbb{C}^n$, it follows from Corollary 2.7 that the Cuntz state is pure.

## 3. The Characterization of Representations of $\mathcal{O}_n$

In this section, we study representations of $\mathcal{O}_n$ and their restrictions to $\mathrm{UHF}_n$.

Let $\pi_1$ and $\pi_2$ be non-degenerate representations of $\mathcal{O}_n$ on a Hilbert space $\mathcal{H}$ and $\pi_1|_{\mathrm{UHF}_n}$ and $\pi_2|_{\mathrm{UHF}_n}$ their restrictions to $\mathrm{UHF}_n$. There exist two unitary operators $U \in \mathcal{B}(\mathcal{H})$ and $M = [m_{ij}] \in M_n(\mathcal{B}(\mathcal{H}))$ such

that

$$T_i = US_i = \sum_{j=1}^{n} S_j m_{ji},$$

where $S_i = \pi_1(s_i)$ and $T_i = \pi_2(s_i)$ for $i = 1, 2, ..., n$. It is known that $U$ and $M$ are given uniquely by $m_{ij} = S_i^* T_j$ and $U = \sum_{j=1}^{n} T_j S_j^*$ (see [3]). In this section, we use the notations described above. We note that a non-degenerate representation of $\mathcal{O}_n$ is injective.

**Theorem 3.1.** *The following conditions are equivalent:*

(1) $\pi_2(\mathcal{O}_n) \subset \pi_1(\mathcal{O}_n)$.

(2) $U \in \pi_1(\mathcal{O}_n)$.

(3) *There is a unitary* $u \in \mathcal{O}_n$ *such that* $\pi_1(u) = U$.

(4) *There is an endomorphism* $\alpha$ *of* $\mathcal{O}_n$ *such that* $\pi_2 = \pi_1 \circ \alpha$.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $\pi_2(\mathcal{O}_n) \subset \pi_1(\mathcal{O}_n)$. Then we have $T_j S_j^* \in \pi_1(\mathcal{O}_n)$ and so $U = \sum_{j=1}^{n} T_j S_j^* \in \pi_1(\mathcal{O}_n)$.

(2) $\Rightarrow$ (3) If $U \in \pi_1(\mathcal{O}_n)$, then there is an element $u \in \mathcal{O}_n$ such that $\pi_1(u) = U$. The facts that $\pi_1(uu^*) = UU^*$, $\pi_1(u^*u) = U^*U$, and $\pi_1(I) = I$ give $uu^* = u^*u = I$.

(3) $\Rightarrow$ (4) For a unitary $u \in \mathcal{O}_n$ with $\pi_1(u) = U$, consider the set $X = \{us_1, us_2, ..., us_n\}$. Since $X$ also satisfies the Cuntz relations, a map $\alpha : \{s_1, s_2, ..., s_n\} \to X$ defined by $\alpha(s_i) = us_i$ can be extended to an endomorphism of $\mathcal{O}_n$ and we denote its extension by $\alpha$. Moreover, we get

$$\pi_2(s_i) = T_i = US_i = \pi_1(us_i) = (\pi_1 \circ \alpha)(s_i)$$

which implies that $\pi_2 = \pi_1 \circ \alpha$.

(4) $\Rightarrow$ (1) It is clear.

**Corollary 3.2.** *The following conditions are equivalent:*

(1) $\pi_2(\mathcal{O}_n) \subset \pi_1(\mathcal{O}_n)$.

(2) $M \in M_n(\pi_1(\mathcal{O}_n))$.

(3) *There is a unitary operator* $[a_{ij}] \in M_n(\mathcal{O}_n)$ *such that* $\pi_1(a_{ij}) = m_{ij}$.

**Proof.** (1) $\Leftrightarrow$ (2) Suppose that $\pi_2(\mathcal{O}_n) \subset \pi_1(\mathcal{O}_n)$ which is equivalent to $U \in \pi_1(\mathcal{O}_n)$ by Theorem 3.1. Then we have $m_{ij} = S_i^*T_j = S_i^*US_j \in \pi_1(\mathcal{O}_n)$ and so $M \in M_n(\pi_1(\mathcal{O}_n))$.

Conversely, suppose that $M = [m_{ij}] \in M_n(\pi_1(\mathcal{O}_n))$ and so $m_{ij} \in \pi_1(\mathcal{O}_n)$. Since $U = \sum_{j=1}^n T_jS_j^* = \sum_{i,\, j=1}^n S_i m_{ij} S_j^*$, we have $U \in \pi_1(\mathcal{O}_n)$ which is equivalent to $\pi_2(\mathcal{O}_n) \subset \pi_1(\mathcal{O}_n)$.

(1) $\Rightarrow$ (3) Suppose that $\pi_2(\mathcal{O}_n) \subset \pi_1(\mathcal{O}_n)$. Then we have $\pi_1(u) = U$ for some unitary $u$ in $\mathcal{O}_n$ by (1) $\Leftrightarrow$ (3) of Theorem 3.1. If we let $a_{ij}$ be $s_i^*us_j$, then we have $a_{ij} \in \mathcal{O}_n$ and

$$\pi_1(a_{ij}) = \pi_1(s_i)^*\pi_1(u)\pi_1(s_j) = S_i^*US_j = m_{ij}.$$

Since $[a_{ij}]^*[a_{ij}] = \left[\sum_{k=1}^n a_{ki}^*a_{kj}\right] = [s_i^*s_j]$ and $[a_{ij}][a_{ij}]^* = \left[\sum_{k=1}^n a_{ik}a_{jk}^*\right] = [s_i^*s_j]$, $[a_{ij}]$ is a unitary operator.

(3) $\Rightarrow$ (1) Suppose that there is a unitary operator $[a_{ij}] \in M_n(\mathcal{O}_n)$ such that $\pi_1(a_{ij}) = m_{ij}$. If we put $u = \sum_{i,\, j=1}^n s_i a_{ij} s_j^*$, then we have $u \in \mathcal{O}_n$ and

$$\pi_1(u) = \sum_{i,\, j=1}^n \pi_1(s_i)\pi_1(a_{ij})\pi_1(s_j^*) = \sum_{i,\, j=1}^n S_i m_{ij} S_j^* = \sum_{j=1}^n T_j S_j^* = U.$$

Since $U$ is a unitary, simple calculations give that $u$ is also a unitary. The proof is completed from (1) $\Leftrightarrow$ (3) of Theorem 3.1.

In Theorem 3.1 and Corollary 3.2, the roles of $\pi_1$ and $\pi_2$ can be interchanged and we obtain the following equivalent conditions for $\pi_1(\mathcal{O}_n) = \pi_2(\mathcal{O}_n)$. The following corollaries are easily verified from Theorem 3.1 and Corollary 3.2. So we omit the details.

**Corollary 3.3.** *The following conditions are equivalent*:

(1) $\pi_1(\mathcal{O}_n) = \pi_2(\mathcal{O}_n)$.

(2) $U \in \pi_1(\mathcal{O}_n) \cap \pi_2(\mathcal{O}_n)$.

(3) $M \in M_n(\pi_1(\mathcal{O}_n)) \cap M_n(\pi_2(\mathcal{O}_n))$.

**Corollary 3.4.** *The following conditions are equivalent*:

(1) $\pi_1(\mathcal{O}_n) = \pi_2(\mathcal{O}_n)$.

(2) *There are unitaries $u$, $v$ in $\mathcal{O}_n$ such that* $\pi_1(u) = \pi_2(v) = U$.

(3) *There are unitary operators* $[a_{ij}], [b_{ij}]$ *in* $M_n(\mathcal{O}_n)$ *such that* $\pi_1(a_{ij}) = \pi_2(b_{ij}) = m_{ij}$.

(4) *There are endomorphisms $\alpha$ and $\beta$ of $\mathcal{O}_n$ such that* $\pi_2 = \pi_1 \circ \alpha$ *and* $\pi_1 = \pi_2 \circ \beta$.

In (4) of Corollary 3.4, $\alpha$ and $\beta$ are actually automorphisms with $\beta = \alpha^{-1}$. The following gives an equivalent condition for $\pi_1|_{\mathrm{UHF}_n} = \pi_2|_{\mathrm{UHF}_n}$.

**Theorem 3.5.** *Let $\pi_1$ and $\pi_2$ be two non-degenerate representations of $\mathcal{O}_n$. Then* $\pi_1|_{\mathrm{UHF}_n} = \pi_2|_{\mathrm{UHF}_n}$ *if and only if* $U \in \pi_1(\mathrm{UHF}_n)'$.

**Proof.** Suppose that $\pi_1|_{\mathrm{UHF}_n} = \pi_2|_{\mathrm{UHF}_n}$. Then we have $T_iT_j^* = S_iS_j^*$ and $US_iS_j^* = S_iS_j^*U$ by $T_iT_j^* = (US_i)(US_j)^*$. Similarly, we get $T_iT_jT_k^*T_l^* = S_iS_jS_k^*S_l^*$ and

$$T_iT_jT_k^*T_l^* = US_iUS_jS_k^*U^*S_l^*U^* = US_iS_jS_k^*S_l^*U^*$$

which imply $US_iS_jS_k^*S_l^* = S_iS_jS_k^*S_l^*U$. In this way, we obtain that for $i_1, ..., i_k, j_1, ..., j_k = 1, 2, ..., n$,

$$US_{i_1}\cdots S_{i_k}S_{j_k}^*\cdots S_{j_1}^* = S_{i_1}\cdots S_{i_k}S_{j_k}^*\cdots S_{j_1}^*U$$

which implies $U \in \pi_1(\mathrm{UHF}_n)'$.

Conversely, suppose that $U \in \pi_1(\mathrm{UHF}_n)'$. Then we get $T_iT_j^* = US_iS_j^*U^* = S_iS_j^*$. By repeating this process, we get $T_iT_jT_k^*T_l^* = S_iS_jS_k^*S_l^*$ and

$$S_{i_1}\cdots S_{i_k}S_{j_k}^*\cdots S_{j_1}^* = T_{i_1}\cdots T_{i_k}T_{j_k}^*\cdots T_{j_1}^*$$

for $i_1, ..., i_k, j_1, ..., j_k = 1, 2, ..., n$. Thus we conclude that $\pi_1|_{\mathrm{UHF}_n} = \pi_2|_{\mathrm{UHF}_n}$.

In the above theorem, $\pi_1|_{\mathrm{UHF}_n} = \pi_2|_{\mathrm{UHF}_n}$ if and only if $U \in \pi_2(\mathrm{UHF}_n)'$. As an easy consequence of Theorem 3.5, we get that $\pi_1|_{\mathrm{UHF}_n} = \pi_2|_{\mathrm{UHF}_n}$ if and only if $U \in \pi_1(\mathrm{UHF}_n)' \cap \pi_2(\mathrm{UHF}_n)'$. Moreover, some manipulations give that $\pi_1|_{\mathrm{UHF}_n} = \pi_2|_{\mathrm{UHF}_n}$ if and only if $U \in \pi_1(\mathrm{UHF}_n)' \cup \pi_2(\mathrm{UHF}_n)'$.

At this point, we briefly comment on the relation between $\pi_1(\mathcal{O}_n) = \pi_2(\mathcal{O}_n)$ and $\pi_1|_{\mathrm{UHF}_n} = \pi_2|_{\mathrm{UHF}_n}$. From our results, we conclude that $\pi_1(\mathcal{O}_n) = \pi_2(\mathcal{O}_n)$ and $\pi_1|_{\mathrm{UHF}_n} = \pi_2|_{\mathrm{UHF}_n}$ are independent. But the following corollary gives a sufficient condition for $\pi_1(\mathcal{O}_n) = \pi_2(\mathcal{O}_n)$ related to their restrictions to $\mathrm{UHF}_n$.

**Corollary 3.6.** *If $\pi_1|_{\mathrm{UHF}_n}$ and $\pi_2|_{\mathrm{UHF}_n}$ are irreducible representations with $\pi_1|_{\mathrm{UHF}_n} = \pi_2|_{\mathrm{UHF}_n}$, then there exists $\lambda \in \mathbb{C}, |\lambda| = 1$ such that $\pi_2(s_i) = \lambda\pi_1(s_i)$ for $i = 1, 2, ..., n$.*

**Proof.** Suppose that $\pi_1|_{\mathrm{UHF}_n}$ and $\pi_2|_{\mathrm{UHF}_n}$ are irreducible representations of $\mathrm{UHF}_n$ with $\pi_1|_{\mathrm{UHF}_n} = \pi_2|_{\mathrm{UHF}_n}$. Then by Theorem 3.5, $U \in \pi_1(\mathrm{UHF}_n)'$. Since $\pi_1|_{\mathrm{UHF}_n}$ is irreducible, $\pi_1(\mathrm{UHF}_n)' = \mathbb{C} \cdot I$ and so $U = \lambda I$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$ which completes the proof.

It is known that there exists a representation $\pi$ of $\mathcal{O}_n$ such that $\mathrm{UHF}_n$ is weakly dense in $\mathcal{B}(\mathcal{H})$ which is equivalent to $\pi|_{\mathrm{UHF}_n}$ is irreducible (see Theorem 3.3 and Theorem 6.8 in [3]).

**Lemma 3.7.** *Let $x$ be an element of $\mathcal{O}_n$. If $xy = yx$ for all $y \in \mathrm{UHF}_n$, then $x = \lambda I$ for some $\lambda \in \mathbb{C}$.*

**Proof.** Take a representation $\pi$ of $\mathcal{O}_n$ with $\pi(\mathrm{UHF}_n)' = \mathbb{C} \cdot I$. For an element $x \in \mathcal{O}_n$ if $x$ satisfies $xy = yx$ for all $y \in \mathrm{UHF}_n$, then we have $\pi(x)\pi(y) = \pi(xy) = \pi(yx) = \pi(y)\pi(x)$ and so $\pi(x) \in \pi(\mathrm{UHF}_n)'$. Hence, we get $\pi(x) = \lambda I$ for some $\lambda \in \mathbb{C}$ and $x = \lambda I$.

**Lemma 3.8.** *For any endomorphism $\alpha$ of $\mathcal{O}_n$, there is a unitary $u$ in $\mathcal{O}_n$ such that $\alpha(s_i) = us_i$ for $i = 1, 2, ..., n$.*

**Proof.** Let $\pi$ be a non-degenerate representation of $\mathcal{O}_n$. Then for an endomorphsim $\alpha$ of $\mathcal{O}_n$, $\pi \circ \alpha$ is also a non-degenerate representation of $\mathcal{O}_n$. Since $(\pi \circ \alpha)(\mathcal{O}_n) \subset \pi(\mathcal{O}_n)$, by Theorem 3.1, there is a unitary $u \in \mathcal{O}_n$ with $\pi(\alpha(s_i)) = \pi(u)\pi(s_i) = \pi(us_i)$ for $i = 1, 2, ..., n$ and we get $\alpha(s_i) = us_i$.

As an application of our result, we now turn our attention to endomorphisms of $\mathcal{O}_n$. Let $End(\mathcal{O}_n)$ (resp. $Inn(\mathcal{O}_n)$) be the set of all endomorphisms (resp. inner automorphisms) of $\mathcal{O}_n$ and $\mathcal{U}(\mathcal{O}_n)$ the set of all unitaries in $\mathcal{O}_n$. At last, we get the following results.

**Theorem 3.9.** (1) $Inn(\mathcal{O}_n) \neq \{id\}$.

(2) *There exists a one-to-one correspondence between $End(\mathcal{O}_n)$ and $\mathcal{U}(\mathcal{O}_n)$.*

**Proof.** (1) Since $\mathcal{U}(\mathcal{O}_n)$ is not trivial, to prove $Inn(\mathcal{O}_n) \neq \{id\}$, it suffices to show that for any $u \in \mathcal{U}(\mathcal{O}_n)$ with $uxu^* = x$ for all $x \in \mathcal{O}_n$, $u$ must be a scalar multiple of $I$. For this, we use Lemma 3.7 which completes the proof.

(2) For a given $\alpha \in End(\mathcal{O}_n)$, from Lemma 3.8, there exists $u_\alpha \in \mathcal{U}(\mathcal{O}_n)$ with $\alpha(s_i) = u_\alpha s_i$. Thus we can define $\Phi : End(\mathcal{O}_n) \to \mathcal{U}(\mathcal{O}_n)$ by $\Phi(\alpha) = u_\alpha$. On the other hand, for any $u \in \mathcal{U}(\mathcal{O}_n)$, there exists an endomorphism $\alpha_u$ of $\mathcal{O}_n$ with $\alpha_u(s_i) = us_i$ by the same argument as in the proof of (3) $\Rightarrow$ (4) in Theorem 3.1. So we can define $\Psi : \mathcal{U}(\mathcal{O}_n) \to End(\mathcal{O}_n)$ by $\Psi(u) = \alpha_u$.

Then for any $\alpha \in End(\mathcal{O}_n)$, we have

$$(\Psi \circ \Phi)(\alpha)(s_i) = \Psi(u_\alpha)(s_i) = u_\alpha s_i = \alpha(s_i)$$

and so $\Psi \circ \Phi(\alpha) = \alpha$ which implies $\Psi \circ \Phi = id$. Moreover, since for any $u \in \mathcal{U}(\mathcal{O}_n)$, $(\Phi \circ \Psi)(u) s_i = \Phi(\alpha_u) s_i = us_i$ and $(\Phi \circ \Psi)(u) s_i s_i^* = us_i s_i^*$ hold, the Cuntz relation $\sum s_i s_i^* = I$ gives $(\Phi \circ \Psi)(u) = u$ which implies $\Phi \circ \Psi = id$.

Before closing this section, let us check a property for two unitaries in $\mathcal{O}_n$ related to an endomorphism of $\mathcal{O}_n$. For any $\alpha \in Inn(\mathcal{O}_n)$, we get two unitaries $v$ and $u$ in $\mathcal{O}_n$ such that $\alpha(x) = vxv^*$ for $x \in \mathcal{O}_n$ and $\alpha(s_i) = us_i$ for $i = 1, 2, ..., n$. In this case, we have

$$us_i s_j^* u^* = (us_i)(us_j)^* = (vs_i v^*)(vs_j v^*)^* = vs_i s_j^* v^*$$

for $i, j = 1, 2, ..., n$ and hence $v^*u$ commutes $s_i s_j^*$. But from Lemma 3.7, we know that $v^*u$ cannot commute all elements in $\mathrm{UHF}_n$.

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# STRONGLY SEMI β-IRRESOLUTE FUNCTIONS AND SEMI α-PREIRRESOLUTE FUNCTIONS

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## Abstract

The concepts of strongly semi β-irresolute functions and semi α-preirresolute functions in topological spaces are introduced and studied. Some of their characteristic properties are considered. Also, we investigate the relationships between these classes of functions and other classes of non-continuous functions.

## 1. Introduction

Recall eight classes of functions called strongly β-irresolute [20], β-irresolute [15], strongly $M$-precontinuous [4], strongly α-irresolute [13], α-irresolute [14], α-continuous [18], semi continuous [12] and β-continuous [1] functions in topological spaces. In 2000, Beceren introduced the concepts of semi α-irresolute functions [5] and almost α-irresolute functions [6]. Recently, Beceren and Noiri have introduced the notions of α-preirresolute functions and β-preirresolute functions [7], and also introduced the concept of strongly α-preirresolute functions [8].

The purpose of the present paper is to introduce and investigate the

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notions of new classes of functions, namely strongly semi β-irresolute functions and semi α-preirresolute functions, and give several characterizations and their properties. Relations between these types of functions and other classes of functions are obtained. The new class of strongly semi β-irresolute functions, which is stronger than β-irresolute functions [15], is a generalization of strongly α-preirresolute functions [8]. The new class of semi α-preirresolute functions, which is stronger than β-preirresolute functions [7], is a generalization of α-preirresolute functions [7].

## 2. Preliminaries

Throughout this note, spaces always mean topological spaces and $f : X \to Y$ denotes a single valued function of a space $X$ into a space $Y$. Let $S$ be a subset of a space $X$. The closure and the interior of $S$ are denoted by $Cl(S)$ and $Int(S)$, respectively.

**Definition 2.1.** A subset $S$ of a space $X$ is said to be *α-open* [22] (resp. *semi open* [12], *preopen* [16], *β-open* [1]) if $S \subset Int(Cl(Int(S)))$ (resp. $S \subset Cl(Int(S))$, $S \subset Int(Cl(S))$, $S \subset Cl(Int(Cl(S)))$).

The family of all α-open (resp. semi open, preopen, β-open) sets in a space $(X, \tau)$ is denoted by $\tau^\alpha$ (resp. $SO(X)$, $PO(X)$, $\beta O(X)$). It is shown in [22] that $\tau^\alpha$ is a topology for $X$. Moreover, $\tau \subset \tau^\alpha \subset SO(X) \subset \beta O(X)$. The complement of an α-open (resp. semi open, preopen, β-open) set is said to be *α-closed* [18] (resp. *semi closed* [9], *preclosed* [16], *β-closed* [1]). The union of all preopen (resp. β-open) sets contained in $S$ is called the *preinterior* [19] (resp. *β-interior* [3]) of $S$ and is denoted by $pInt(S)$ (resp. $\beta Int(S)$); the intersection of all preclosed (resp. β-closed) sets containing a subset $S$ is called the *preclosure* [19] (resp. *β-closure* [3]) of $S$ and is denoted by $pCl(S)$ (resp. $\beta Cl(S)$).

**Definition 2.2.** A function $f : (X, \tau) \to (Y, \upsilon)$ is said to be *α-continuous* [18] (resp. *semi continuous* [12], *β-continuous* [1]) if $f^{-1}(V)$ is α-open (resp. semi open, β-open) in $X$ for every open set $V$ of $Y$.

**Definition 2.3.** A function $f : (X, \tau) \to (Y, \upsilon)$ is said to be *strongly α-irresolute* [13] (resp. *α-irresolute* [14], *semi α-irresolute* [5], *almost α-irresolute* [6]) if $f^{-1}(V)$ is open (resp. α-open, semi open, β-open) in $X$ for every α-open set $V$ of $Y$.

**Definition 2.4.** A function $f : (X, \tau) \to (Y, \upsilon)$ is said to be *strongly M-precontinuous* [4] (resp. *α-preirresolute* [7], *β-preirresolute* [7]) if $f^{-1}(V)$ is open (resp. α-open, β-open) in $X$ for every preopen set $V$ of $Y$.

**Definition 2.5.** A function $f : (X, \tau) \to (Y, \upsilon)$ is said to be *strongly β-irresolute* [20] (resp. *strongly α-preirresolute* [8], *β-irresolute* [15]) if $f^{-1}(V)$ is open (resp. α-open, β-open) in $X$ for every β-open set $V$ of $Y$.

**Definition 2.6.** A function $f : (X, \tau) \to (Y, \upsilon)$ is said to be *strongly semi β-irresolute* (resp. *semi α-preirresolute*) if $f^{-1}(V)$ is semi open in $X$ for every β-open (resp. preopen) set $V$ of $Y$.

From the definitions stated above, we obtain the following diagram:

$$\begin{tikzcd}
\textit{Strongly } \beta\textit{-irresoluteness} \arrow[r] \arrow[d] & \textit{strongly } \alpha\textit{-preirresoluteness} \arrow[r] \arrow[d] & \textit{strongly semi } \beta\textit{-irresoluteness} \arrow[r] \arrow[d] & \beta\textit{-irresoluteness} \arrow[d] \\
\textit{Strongly M-precontinuity} \arrow[r] \arrow[d] & \alpha\textit{-preirresoluteness} \arrow[r] \arrow[d] & \textit{semi } \alpha\textit{-preirresoluteness} \arrow[r] \arrow[d] & \beta\textit{-preirresoluteness} \arrow[d] \\
\textit{Strongly } \alpha\textit{-irresoluteness} \arrow[r] \arrow[d] & \alpha\textit{-irresoluteness} \arrow[r] \arrow[d] & \textit{semi } \alpha\textit{-irresoluteness} \arrow[r] \arrow[d] & \textit{almost } \alpha\textit{-irresoluteness} \arrow[d] \\
\textit{continuity} \arrow[r] & \alpha\textit{-continuity} \arrow[r] & \textit{semi continuity} \arrow[r] & \beta\textit{-continuity}
\end{tikzcd}$$

The following Example and Remark enable us to realize that none of the above implications is reversible.

**Remark.** We have the following relationships:

(a) Strong $M$-precontinuity does not imply β-irresoluteness ([6, Example 3.2]).

(b) Strong α-irresoluteness does not imply β-preirresoluteness ([7, Example 2.2]).

(c) Continuity does not imply almost α-irresoluteness ([6, Example 3.3]).

(d) Strong α-preirresoluteness does not imply continuity ([7, Example 2.1]).

(e) β-irresoluteness does not imply semi continuity ([6, Example 3.1]).

**Example.** Let $X = \{x, y, z\}$ and $\tau = \{X, \varnothing, \{y\}, \{z\}, \{y, z\}\}$. Let $f : (X, \tau) \to (X, \tau)$ be a function defined as follows: $f(x) = f(y) = y$ and $f(z) = z$. Then $f$ is strongly semi β-irresolute, but it is not α-continuous.

## 3. Strongly Semi β-irresolute Functions

**Theorem 3.1.** *For a function $f : (X, \tau) \to (Y, \upsilon)$, the following are equivalent:*

(1) *$f$ is strongly semi β-irresolute;*

(2) *For each $x \in X$ and each β-open set $V$ of $Y$ containing $f(x)$, there exists a semi open set $U$ of $X$ containing $x$ such that $f(U) \subset V$;*

(3) *$f^{-1}(V) \subset Cl(Int(f^{-1}(V)))$ for every β-open set $V$ of $Y$;*

(4) *$f^{-1}(F)$ is semi closed in $X$ for every β-closed set $F$ of $Y$;*

(5) *$Int(Cl(f^{-1}(B))) \subset f^{-1}(\beta Cl(B))$ for every subset $B$ of $Y$;*

(6) *$f(Int(Cl(A))) \subset \beta Cl(f(A))$ for every subset $A$ of $X$.*

**Proof.** (1) ⇒ (2) Let $x \in X$ and let $V$ be any β-open set of $Y$ containing $f(x)$. By Definition 2.6, $f^{-1}(V)$ is semi open in $X$ and contains $x$. Set $U = f^{-1}(V)$, then by (1), $U$ is a semi open subset of $X$ containing $x$ and $f(U) \subset V$.

(2) ⇒ (3) Let $V$ be any β-open set of $Y$ and $x \in f^{-1}(V)$. By (2), there exists a semi open set $U$ of $X$ containing $x$ such that $f(U) \subset V$. Thus, we have $x \in U \subset Cl(Int(U)) \subset Cl(Int(f^{-1}(V)))$ and hence $f^{-1}(V) \subset Cl(Int(f^{-1}(V)))$.

(3) ⇒ (4) Let $F$ be any β-closed subset of $Y$. Set $V = Y - F$, then $V$ is β-open in $Y$. By (3), we obtain $f^{-1}(V) \subset Cl(Int(f^{-1}(V)))$ and hence $f^{-1}(F) = X - f^{-1}(Y - F) = X - f^{-1}(V)$ is semi closed in $X$.

(4) $\Rightarrow$ (5) Let $B$ be any subset of $Y$. Since $\beta Cl(B)$ is a β-closed subset of $Y$, $f^{-1}(\beta Cl(B))$ is semi closed in $X$ and hence $Int(Cl(f^{-1}(\beta Cl(B))))$ $\subset f^{-1}(\beta Cl(B))$. Therefore, we obtain $Int(Cl(f^{-1}(B))) \subset f^{-1}(\beta Cl(B))$.

(5) $\Rightarrow$ (6) Let $A$ be any subset of $X$. By (5), we have $Int(Cl(A)) \subset Int(Cl(f^{-1}(f(A)))) \subset f^{-1}(\beta Cl(f(A)))$ and hence $f(Int(Cl(A))) \subset \beta Cl(f(A))$.

(6) $\Rightarrow$ (1) Let $V$ be any β-open subset of $Y$. Since $f^{-1}(Y - V) = X - f^{-1}(V)$ is a subset of $X$ and by (6), we obtain

$$\begin{aligned} f(Int(Cl(f^{-1}(Y - V)))) &\subset \beta Cl(f(f^{-1}(Y - V))) \\ &\subset \beta Cl(Y - V) \\ &= Y - \beta Int(V) \\ &= Y - V \end{aligned}$$

and hence

$$\begin{aligned} X - Cl(Int(f^{-1}(V))) &= Int(Cl(X - f^{-1}(V))) \\ &= Int(Cl(f^{-1}(Y - V))) \\ &\subset f^{-1}(f(Int(Cl(f^{-1}(Y - V))))) \\ &\subset f^{-1}(Y - V) \\ &= X - f^{-1}(V). \end{aligned}$$

Therefore, we have $f^{-1}(V) \subset Cl(Int(f^{-1}(V)))$ and hence $f^{-1}(V)$ is semi open in $X$. Thus, $f$ is strongly semi β-irresolute.

**Lemma 3.1** ([2], [23], [10]). *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of spaces and $U_{\lambda_i}$ be a nonempty subset of $X_{\lambda_i}$ for each $i = 1, 2, ..., n$. Then $U = \prod_{\lambda \neq \lambda_i} X_\lambda \times \prod_{i=1}^{n} U_{\lambda_i}$ is a nonempty β-open [2] (resp. semi open [23], preopen [10]) subset of $\prod X_\lambda$ if and only if $U_{\lambda_i}$ is β-open (resp. semi open, preopen) in $X_{\lambda_i}$ for each $i = 1, 2, ..., n$.*

**Theorem 3.2.** *A function $f : X \to Y$ is strongly semi $\beta$-irresolute if the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is strongly semi $\beta$-irresolute.*

**Proof.** Let $x \in X$ and $V$ be any $\beta$-open set of $Y$ containing $f(x)$. Then $X \times V$ is a $\beta$-open set of $X \times Y$ by Lemma 3.1 and contains $g(x)$. Since $g$ is strongly semi $\beta$-irresolute, there exists a semi open set $U$ of $X$ containing $x$ such that $g(U) \subset X \times V$ and hence $f(U) \subset V$. Thus $f$ is strongly semi $\beta$-irresolute.

**Theorem 3.3.** *If a function $f : X \to \prod Y_\lambda$ is strongly semi $\beta$-irresolute, then $P_\lambda \circ f : X \to Y_\lambda$ is strongly semi $\beta$-irresolute for each $\lambda \in \Lambda$, where $P_\lambda$ is the projection of $\prod Y_\lambda$ onto $Y_\lambda$.*

**Proof.** Let $V_\lambda$ be any $\beta$-open set of $Y_\lambda$. Since $P_\lambda$ is continuous and open, it is $\beta$-irresolute [1, Theorem 2.2], and hence $P_\lambda^{-1}(V_\lambda)$ is $\beta$-open in $\prod Y_\lambda$. Since $f$ is strongly semi $\beta$-irresolute, $f^{-1}(P_\lambda^{-1}(V_\lambda)) = (P_\lambda \circ f)^{-1}(V_\lambda)$ is semi open in $X$. Hence $P_\lambda \circ f$ is strongly semi $\beta$-irresolute for each $\lambda \in \Lambda$.

**Theorem 3.4.** *If the product function $f : \prod X_\lambda \to \prod Y_\lambda$ is strongly semi $\beta$-irresolute, then $f_\lambda : X_\lambda \to Y_\lambda$ is strongly semi $\beta$-irresolute for each $\lambda \in \Lambda$.*

**Proof.** Let $\lambda_0 \in \Lambda$ be an arbitrary fixed index and $V_{\lambda_0}$ be any $\beta$-open set of $Y_{\lambda_0}$. Then, $\prod Y_\gamma \times V_{\lambda_0}$ is $\beta$-open in $\prod Y_\lambda$ by Lemma 3.1, where $\lambda_0 \neq \gamma \in \Lambda$. Since $f$ is strongly semi $\beta$-irresolute, $f^{-1}\left(\prod Y_\gamma \times V_{\lambda_0}\right) = \prod X_\gamma \times f_{\lambda_0}^{-1}(V_{\lambda_0})$ is semi open in $\prod X_\lambda$ and hence, by Lemma 3.1, $f_{\lambda_0}^{-1}(V_{\lambda_0})$ is semi open in $X_{\lambda_0}$. This implies that $f_{\lambda_0}$ is strongly semi $\beta$-irresolute.

**Theorem 3.5.** *If $f : (X, \tau) \to (Y, \upsilon)$ is strongly semi β-irresolute and A is a preopen subset of X, then the restriction $f/A : A \to Y$ is strongly semi β-irresolute.*

**Proof.** Let $V$ be any β-open set of $Y$. Since $f$ is strongly semi β-irresolute, $f^{-1}(V)$ is semi open in $X$. Since $A$ is preopen in $X$, by Lemma 1.1 of [17], $(f/A)^{-1}(V) = A \cap f^{-1}(V)$ is semi open in $A$ and hence $f/A$ is strongly semi β-irresolute.

**Theorem 3.6.** *Let $f : (X, \tau) \to (Y, \upsilon)$ be a function and $\{A_\lambda : \lambda \in \Lambda\}$ be a cover of X by semi open sets of $(X, \tau)$. Then f is strongly semi β-irresolute if $f/A_\lambda : A_\lambda \to Y$ is strongly semi β-irresolute for each $\lambda \in \Lambda$.*

**Proof.** Let $V$ be any β-open set of $Y$. Since $f/A_\lambda$ is strongly semi β-irresolute, $(f/A_\lambda)^{-1}(V)$ is semi open in $A_\lambda$. Since $A_\lambda$ is semi open in $X$, by Theorem 5 of [23], $(f/A_\lambda)^{-1}(V)$ is semi open in $X$ for each $\lambda \in \Lambda$. Therefore,

$$
\begin{aligned}
f^{-1}(V) &= X \cap f^{-1}(V) \\
&= \bigcup \{A_\lambda \cap f^{-1}(V) : \lambda \in \Lambda\} \\
&= \bigcup \{(f/A_\lambda)^{-1}(V) : \lambda \in \Lambda\}
\end{aligned}
$$

is semi open in $X$ because the union of semi open sets is a semi open set. Hence $f$ is strongly semi β-irresolute.

**Theorem 3.7.** *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the composition $g \circ f : X \to Z$ is strongly semi β-irresolute if f is strongly semi β-irresolute and g is β-irresolute.*

**Proof.** Let $W$ be any β-open subset of $Z$. Since $g$ is β-irresolute, $g^{-1}(W)$ is β-open in $Y$. Since $f$ is strongly semi β-irresolute, $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ is semi open in $X$ and hence $g \circ f$ is strongly semi β-irresolute.

## 4. Semi $\alpha$-preirresolute Functions

**Theorem 4.1.** *For a function* $f : (X, \tau) \to (Y, \upsilon)$, *the following are equivalent*:

(1) *$f$ is semi $\alpha$-preirresolute;*

(2) *For each $x \in X$ and each preopen set $V$ of $Y$ containing $f(x)$, there exists a semi open set $U$ of $X$ containing $x$ such that $f(U) \subset V$;*

(3) $f^{-1}(V) \subset Cl(Int(f^{-1}(V)))$ *for every preopen set $V$ of $Y$;*

(4) $f^{-1}(F)$ *is semi closed in $X$ for every preclosed set $F$ of $Y$;*

(5) $Int(Cl(f^{-1}(B))) \subset f^{-1}(pCl(B))$ *for every subset $B$ of $Y$;*

(6) $f(Int(Cl(A))) \subset pCl(f(A))$ *for every subset $A$ of $X$.*

**Proof.** The proof is similar to that of Theorem 3.1 and is thus omitted.

**Theorem 4.2.** *A function $f : X \to Y$ is semi $\alpha$-preirresolute if the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is semi $\alpha$-preirresolute.*

**Proof.** This follows from Lemma 3.1 and Theorem 3.2.

**Theorem 4.3.** *If a function $f : X \to \prod Y_\lambda$ is semi $\alpha$-preirresolute, then $P_\lambda \circ f : X \to Y_\lambda$ is semi $\alpha$-preirresolute for each $\lambda \in \Lambda$, where $P_\lambda$ is the projection of $\prod Y_\lambda$ onto $Y_\lambda$.*

**Proof.** This follows immediately from Theorem 3.4 in [18] and Theorem 3.3.

**Theorem 4.4.** *If the product function $f : \prod X_\lambda \to \prod Y_\lambda$ is semi $\alpha$-preirresolute, then $f_\lambda : X_\lambda \to Y_\lambda$ is semi $\alpha$-preirresolute for each $\lambda \in \Lambda$.*

**Proof.** This follows from Lemma 3.1 and Theorem 3.4.

**Theorem 4.5.** *If $f : (X, \tau) \to (Y, \upsilon)$ is semi α-preirresolute and A is a preopen subset of X, then the restriction $f/A : A \to Y$ is semi α-preirresolute.*

**Proof.** The proof is similar to that of Theorem 3.5 and is thus omitted.

**Theorem 4.6.** *Let $f : (X, \tau) \to (Y, \upsilon)$ be a function and $\{A_\lambda : \lambda \in \Lambda\}$ be a cover of X by semi open sets of $(X, \tau)$. Then f is semi α-preirresolute if $f/A_\lambda : A_\lambda \to Y$ is semi α-preirresolute for each $\lambda \in \Lambda$.*

**Proof.** The proof is similar to that of Theorem 3.6 and is thus omitted.

**Theorem 4.7.** *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the composition $g \circ f : X \to Z$ is semi α-preirresolute if f is semi α-preirresolute and g is preirresolute.*

**Proof.** Let $W$ be any preopen subset of $Z$. Since $g$ is preirresolute, $g^{-1}(W)$ is preopen in $Y$. Since $f$ is semi α-preirresolute, $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ is semi open in $X$ and hence $g \circ f$ is semi α-preirresolute.

We recall that a space $X$ is said to be *submaximal* if every dense subset of $X$ is open in $X$ and *extremally disconnected* if the closure of each open subset of $X$ is open in $X$. The following two Theorems follow from the fact that if $(X, \tau)$ is a submaximal and extremally disconnected space, then $\tau = \tau^\alpha = SO(X) = PO(X) = \beta O(X)$ (Jankovic [11] and Nasef et al. [21]).

**Theorem 4.8.** *Let $(X, \tau)$ be a submaximal and extremally disconnected space. For a function $f : (X, \tau) \to (Y, \upsilon)$, we have*

*(1) Strongly semi β-irresoluteness $\Leftrightarrow$ strongly β-irresoluteness $\Leftrightarrow$ strongly α-preirresoluteness $\Leftrightarrow$ β-irresoluteness.*

*(2) Semi α-preirresoluteness $\Leftrightarrow$ strongly M-precontinuity $\Leftrightarrow$ α-preirresoluteness $\Leftrightarrow$ β-preirresoluteness.*

(3) *Semi* $\alpha$*-irresoluteness* $\Leftrightarrow$ *strongly* $\alpha$*-irresoluteness* $\Leftrightarrow$ $\alpha$*-irresoluteness* $\Leftrightarrow$ *almost* $\alpha$*-irresoluteness.*

(4) *Continuity* $\Leftrightarrow$ $\alpha$*-continuity* $\Leftrightarrow$ *semi continuity* $\Leftrightarrow$ $\beta$*-continuity.*

**Theorem 4.9.** *Let* $(Y, \upsilon)$ *be a submaximal and extremally disconnected space. Then, for a function* $f : (X, \tau) \to (Y, \upsilon)$*, we have*

(1) *Strongly semi* $\beta$*-irresoluteness* $\Leftrightarrow$ *semi* $\alpha$*-preirresoluteness* $\Leftrightarrow$ *semi* $\alpha$*-irresoluteness* $\Leftrightarrow$ *semi continuity.*

(2) $\beta$*-preirresoluteness* $\Leftrightarrow$ *almost* $\alpha$*-irresoluteness* $\Leftrightarrow$ $\beta$*-irresoluteness* $\Leftrightarrow$ $\beta$*-continuity.*

(3) *Strongly* $\alpha$*-preirresoluteness* $\Leftrightarrow$ $\alpha$*-preirresoluteness* $\Leftrightarrow$ $\alpha$*-irresoluteness* $\Leftrightarrow$ $\alpha$*-continuity.*

(4) *Strongly* $\beta$*-irresoluteness* $\Leftrightarrow$ *strongly M-precontinuity* $\Leftrightarrow$ *strongly* $\alpha$*-irresoluteness* $\Leftrightarrow$ *continuity.*

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# ON THE ELLIPTIC CURVE $(X + Y + Z)^3 = \alpha XYZ$ (I)

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and

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( Received May 24, 2001 )

Submitted by K. K. Azad

## Abstract

We consider the elliptic curves $(X + Y + Z)^3 = \alpha XYZ$ defined over Q. For such $p = 5, 7, 11, 13, 17$ we find solutions. Also, we obtain the conditon such that the torsion group is $\mathbb{Z}/3\mathbb{Z}$, and we classify torsion groups for integers $\alpha$ between $-20$ and 500.

## 1. Introduction

Elliptic curves are curves in the projective plane defined by non-singular cubic equations. From the Mordell-Weil theorem ([1], [2], [6], [8], [10]), we see that the Mordell-Weil group $E(K)$ has the form $E(K) \cong E_{tors}(K) \times \mathbb{Z}^r$, where the torsion subgroup $E_{tors}(K)$ is finite and the rank $r$ of $E(K)$ is a non-negative integer. Let $E_\alpha : y^2 = x^3 - 27(\alpha^4 - 24\alpha^3)x - 54(-\alpha^6 + 36\alpha^5 - 216\alpha^4)$ be a curve with $\alpha \in \mathbb{Z}$. In the next part of this paper, we shall divide into cusp, node, elliptic curve for every integer

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$\alpha$ and find the torsion subgroup, $E_{\alpha,\, tors}(\mathbb{Q})$, of $E(\mathbb{Q})$ by Lutz-Nagell theorem ([3], [6], [7]). We find out if $\alpha \neq 0,\ 27$, then the exact order is 3, $P = (3\alpha^2,\ 2^2 3^3 \alpha^2)$ in $E_{\alpha,\, tors}(\mathbb{Q})$, so we see that the order of $E_{\alpha,\, tors}(\mathbb{Q})$ is divisible by 3 for integer $\alpha \neq 0,\ 27$ (Lemma 2.1, Corollary 2.2). Using the reduction map modulo $p$, Corollary 2.2 and Mazur theorem ([2], [4], [5], [6], [10]), we can classify the torsion group $E_{\alpha,\, tors}(\mathbb{Q})$. Finally, using MATHEMATICA 4.0, we will compute the torsion group of $E_\alpha(\mathbb{Q})$ with $-20 \leq \alpha \leq 500$ (Remark 2.7).

## 2. Torsion Subgroups

Let $E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ denote an elliptic curve in generalized Weierstrass normal form defined over an algebraic number field $K$. Following Tate ([7], [8], [9]), we put $b_2 = a_1^2 + 4a_2$, $b_4 = a_1a_3 + 2a_4$, $b_6 = a_3^2 + 4a_6$, $b_8 = a_1^2a_6 - a_1a_3a_4 + 4a_2a_6 + a_2a_3^2 - a_4^2$, $c_4 = b_2^2 - 24b_4$, and $c_6 = -b_2^3 + 36b_2b_4 - 216b_6$. Then the discriminant of $E$ over $K$ is

$$\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6. \tag{2.1}$$

Assume that $K = \mathbb{Q}$. Let the Diophantine equation

$$(X + Y + Z)^3 = \alpha XYZ \tag{2.2}$$

be a curve defined over $\mathbb{Q}$. By easy substitution, we will write the Weierstrass equation $E_\alpha : y^2 = x^3 - 27c_4x - 54c_6$, where $c_4 = \alpha^4 - 24\alpha^3$ and $c_6 = -\alpha^6 + 36\alpha^5 - 216\alpha^4$. And by the definition of discriminant (2.1) of $E_\alpha$, $\Delta(E_\alpha) = 2^{12}3^{12}\alpha^8(\alpha - 27)$. Then $E_\alpha$ is a cusp (resp., node, elliptic curve) if $\alpha = 0$ (resp., $\alpha = 27$, $\alpha \neq 0,\ 27$) with $\alpha \in \mathbb{Z}$.

**Lemma 2.1.** *Let* $E_\alpha : y^2 = x^3 - 27(\alpha^4 - 24\alpha^3)x - 54(-\alpha^6 + 36\alpha^5 - 216\alpha^4)$ *be an elliptic curve over* $\mathbb{Q}$. *Then* $E_{\alpha,\, tors}(\mathbb{Q})$ *is a non-trivial set. In particular, if* $\alpha\ (\neq 0,\ 27)$ *is an integer, then*

$$P = (3\alpha^2, 2^2 3^3 \alpha^2) \in E_{\alpha, tors}(\mathbb{Q}).$$

**Proof.** Let $f_\alpha(x) = x^3 - 27(\alpha^4 - 24\alpha^3)x - 54(-\alpha^6 + 36\alpha^5 - 216\alpha^4)$ be a cubic polynomial with $\alpha \in \mathbb{Z}$. Since $f_\alpha(3\alpha^2) = 11664\alpha^4 = (\pm 2^2 3^3 \alpha^2)^2$, we see that $3\alpha^2$ is a square of $f_\alpha(x)$. Similarly, so is $f_{-\alpha}(3\alpha^2) = (\pm 2^2 3^3 \alpha^2)^2$. There exist points $(3\alpha^2, \pm 2^2 3^3 \alpha^2)$ in $E_\alpha(\mathbb{Q})$ and $E_{-\alpha}(\mathbb{Q})$ for $\alpha$. Now put $X = x(P)$ and use the duplication formula ([6]): $\phi(X) = X^4 - 2aX^2 - 8bX + a^2$, $\psi(X) = X^3 + aX + b$, where $a = -27(\alpha^4 - 24\alpha^3)$, $b = -54(-\alpha^6 + 36\alpha^5 - 216\alpha^4)$, $x([2]P) = \dfrac{\phi(X)}{4\psi(X)}$. So, we get $x(P) = 3\alpha^2 = x(2P)$. Since $P \neq 2P$, $3P = O$. Hence $E_\alpha(\mathbb{Q})$ and $E_{-\alpha}(\mathbb{Q})$ are non-trivial sets.

**Corollary 2.2.** *The order of $E_{\alpha, tors}(\mathbb{Q})$ is divided by 3 for integer $\alpha (\neq 0, 27)$.*

**Proof.** Since $\{O, P, -P\}$ is a subgroup of $E_{\alpha, tors}(\mathbb{Q})$, $3 \,\big|\, |E_{\alpha, tors}(\mathbb{Q})|$.

**Lemma 2.3.** *Let $f(x) = x^3 - 27(\alpha^4 - 24\alpha^3)x - 54(-\alpha^6 + 36\alpha^5 - 216\alpha^4)$ and let $\beta, \gamma, \delta$ be roots of $f(x)$. Then, we get*

(a) *$f(x)$ has one real root and two complex roots if $\alpha < 27$ and $\alpha \neq 0$.*

(b) *$f(x)$ has three real roots if $\alpha \geq 27$ or $\alpha = 0$.*

**Proof.** The discriminant of $E_\alpha$ is $\Delta(E_\alpha) = 2^{12} \cdot 3^{12} \cdot \alpha^8(\alpha - 27)$. We deduce from (2.1) that

$$\Delta = 16(\delta - \beta)^2(\delta - \gamma)^2(\gamma - \beta)^2. \tag{2.3}$$

Also, since the degree of $f(x)$ is 3, $f(x)$ have either one real and two complex roots or three real roots. Assume $\delta \in \mathbb{R}$, $\beta = x + iy$ and $\gamma = x - iy$ with $x, y \in \mathbb{R}$. By (2.3), we derived that $\Delta = -64y^2(x^2 + y^2 - 2x\delta + \delta^2)^2$.

Since $\alpha < 27$ and $\alpha \neq 0$, so $D < 0$. It follows that $y \neq 0$. If $\alpha = 0$, then $f(x) = x^3$ have triple real roots. If $\alpha = 27$, then $f(x) = x^3 - 1594323x - 774840978 = (x + 729)^2(x - 1458)$. $f(x)$ has three real roots and multiple roots exist, so $f(x)$ is a node. If $\alpha > 27$, then $\Delta > 0$ and $f(x)$ has three real roots.

**Corollary 2.4.** *If* $\alpha > 27$, *then* $12 \mid |E_{\alpha, tors}(\mathbb{R})|$. *And, if* $\alpha < 27$, *then* $6 \mid |E_{\alpha, tors}(\mathbb{R})|$. *Furthermore, if* $E_{\alpha, tors}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$, *then* $\alpha > 27$.

**Proof.** Let

$$\begin{aligned} f(x) &= x^3 - 27(\alpha^4 - 24\alpha^3)x - 54(-\alpha^6 + 36\alpha^5 - 216\alpha^4) \\ &= (x - \beta_1)(x - \beta_2)(x - \beta_3). \end{aligned}$$

By Theorem 2.3, we get

$$\begin{cases} \beta_1, \beta_2, \beta_3 \in \mathbb{R} & \text{if } \alpha > 27 \\ \beta_1 \in \mathbb{R}, \beta_2, \beta_3 \in \mathbb{C} - \mathbb{R} & \text{if } \alpha < 27. \end{cases}$$

Assume that $\alpha \neq 0, 27$ is an integer, then the Weierstrass equation

$$E_\alpha : y^2 = x^3 - 27(\alpha^4 - 24\alpha^3)x - 54(-\alpha^6 + 36\alpha^5 - 216\alpha^4)$$

is an elliptic curve defined over $\mathbb{Q}$. Then we consider the reduction $\widetilde{E}_\alpha$ of $E_\alpha$ modulo $p$, where $p$ is prime which does not divide $\Delta$ of each $E_\alpha$, where $\Delta = -16(4a^3 + 27b^2)$. So, we get the result.

**Proposition 2.5** ([6]). *Let* $E/\mathbb{Q}$ *be an elliptic curve and* $m \geq 1$ *be an integer relatively prime to* $char(\mathbb{F}_p)$. *If the reduced curve* $\widetilde{E}/\mathbb{F}_p$ *is non-singular, then the reduction map* $E(\mathbb{Q})[m] \to \widetilde{E}/\mathbb{F}_p$ *is injective. (Here* $E(\mathbb{Q})[m]$ *denotes the set of points of order* $m$ *in* $E(\mathbb{Q})$.*)*

Using Proposition 2.5 and Corollary 2.2, the possibility of $E_{\alpha, tors}(\mathbb{Q})$ are $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/9\mathbb{Z}$, $\mathbb{Z}/12\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$. First, we can

compute the solutions of $\widetilde{E}_\alpha/\mathbb{F}_5$, $\widetilde{E}_\alpha/\mathbb{F}_7$, $\widetilde{E}_\alpha/\mathbb{F}_{11}$, $\widetilde{E}_\alpha/\mathbb{F}_{13}$ (see the Appendix tables). Using these tables (Appendix) and Proposition 2.5, we get the theorem.

**Theorem 2.6.** *Let* $a \equiv b(m)$ *denoted by* $m \mid (a - b)$. *If* $\alpha \equiv 3(5)$, 3, 6(11), 4, 10(13), 1, 8, 9, 11, 12, 14(17), 11, 16, 17, 22, 26(35), 1, 11, 21, 26, 42, 46, 52(55), 6, 35, 41, 57, 61(65), 1, 9, 10, 15, 19, 22, 24, 30, 43, 45, 53, 57, 59, 66, 74, 75(77), 6, 16, 21, 36, 66, 81(85), 5, 11, 15, 18, 22, 37, 44, 45, 71, 80, 87, 89(91), 15, 36, 38, 50, 57, 64, 66, 87, 101, 106, 108, 115(119), 9, 11, 19, 24, 37, 41, 44, 57, 70, 74, 76, 89, 97, 109, 119, 122(143), *and* 19, 30, 53, 64, 74, 108, 118, 140, 151, 152, 174, 185(187), *then* $E_{\alpha,\, tors}(\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$.

**Remark 2.7.** We compute the torsion group $E_{\alpha,\, tors}(\mathbb{Q})$ $(20 \le \alpha \le 500)$ (Use MATHEMATICA 4.0).

$$\begin{cases} \bullet\ E_{\alpha,\, tors}(\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z} & \text{if } \alpha \neq -1, 0, 2, 27, 32, 54, 125, \\ \bullet\ E_{\alpha,\, tors}(\mathbb{Q}) \cong \mathbb{Z}/6\mathbb{Z} & \text{if } \alpha = -1, 2, 32, 54, 125, \end{cases}$$

i.e.,

$$E_{-1,\, tors}(\mathbb{Q}) = \{O, (-33, 0), (3, \pm 108), (39, \pm 216)\},$$

$$E_{2,\, tors}(\mathbb{Q}) = \{O, (-24, 0), (12, \pm 432), (156, \pm 2160)\},$$

$$E_{32,\, tors}(\mathbb{Q}) = \{O, (-1536, \pm 110592), (768, 0), (3072, \pm 110592)\},$$

$$E_{54,\, tors}(\mathbb{Q}) = \{O, (5832, 0), (8748, \pm 314928), (-2916, \pm 944784)\},$$

and

$$E_{125,\, tors}(\mathbb{Q}) = \{O, (41250, 0), (46875, \pm 1687500), (1875, \pm 11812500)\}.$$

## 3. Appendix

| $\alpha$ | $\widetilde{E}_\alpha/\mathrm{F}_5$ | Point | Torsion Group |
|---|---|---|---|
| 0 | $x^3$ | $\{O, (-1, \pm 2), (0, 0), (1, \pm 1)\}$ | cusp |
| 1 | $x^3+x+4$ | $\{O, (-2, \pm 2), (0, \pm 2), (1, \pm 1), (2, \pm 2)\}$ | $Z/9Z$ |
| 2 | $x^3+2x+2$ | $\{O, (-2, 0), (-1, \pm 2), (1, 0), (2, \pm 2)\}$ | node |
| 3 | $x^3+4x+3$ | $\{O, (2, \pm 2)\}$ | $Z/3Z$ |
| 4 | $x^3+2$ | $\{O, (-2, \pm 2), (-1, \pm 1), (2, 0)\}$ | $Z/6Z$ |

| $\alpha$ | $\widetilde{E}_\alpha/\mathrm{F}_7$ | Point | Group |
|---|---|---|---|
| 0 | $x^3$ | $\{O, (-3, \pm 1), (0, 0), (1, \pm 1), (2, \pm 1)\}$ | cusp |
| 1 | $x^3+5x+2$ | $\{O, (-3, \pm 3), (0, \pm 3), (1, \pm 1), (3, \pm 3)\}$ | $Z/9Z$ |
| 2 | $x^3+6x+3$ | $\{O, (-3, 0), (-2, \pm 2), (2, \pm 3)\}$ | $Z/6Z$ |
| 3 | $x^3+2$ | $\{O, (-2, \pm 1), (-1, \pm 1), (0, \pm 3), (3, \pm 1)\}$ | $Z/9Z$ |
| 4 | $x^3+x+3$ | $\{O, (-3, \pm 1), (-2, 0), (-1, \pm 1)\}$ | $Z/6Z$ |
| 5 | $x^3+5x+1$ | $\{O, (-3, \pm 1), (-2, \pm 2), (-1, \pm 3), (0, \pm 1), (3, \pm 1), (1, 0)\}$ | $Z/12Z$ |
| 6 | $x^3+4x+5$ | $\{O, (-3, \pm 1), (-1, 0), (2, 0), (3, \pm 3)\}$ | node |

| $\alpha$ | $\widetilde{E}_\alpha/\mathbb{F}_{11}$ | Point | Group |
|---|---|---|---|
| 0 | $x^3$ | $\{O, (-2, \pm 5), (0, 0), (1, \pm 1), (3, \pm 4), (4, \pm 3), (5, \pm 2)\}$ | cusp |
| 1 | $x^3 + 5x + 6$ | $\{O, (-1, 0), (1, \pm 1), (3, \pm 2)\}$ | Z/6Z |
| 2 | $x^3 + 8$ | $\{O, (-5, \pm 2), (-3, \pm 5), (-2, 0), (1, \pm 3), (2, \pm 4), (5, \pm 1)\}$ | Z/12Z |
| 3 | $x^3 + 8x + 5$ | $\{O, (-5, \pm 4), (-3, \pm 3), (-2, \pm 5), (0, \pm 4), (1, \pm 5), (3, \pm 1), (5, \pm 4)\}$ | Z/15Z |
| 4 | $x^3 + 9x$ | $\{O, (-3, \pm 1), (-1, \pm 1), (0, 0), (2, \pm 2), (4, \pm 1), (5, \pm 4)\}$ | Z/12Z |
| 5 | $x^3 + 6x + 1$ | $\{O, (-5, 0), (-4, \pm 1), (-3, 0), (-2, \pm 5), (-1, \pm 4), (0, \pm 1), (4, \pm 1)\}$ | node |
| 6 | $x^3 + 3x + 6$ | $\{O, (-5, \pm 3), (-3, \pm 5), (-2, \pm 5), (2, \pm 3), (3, \pm 3), (4, \pm 4), (5, \pm 5)\}$ | Z/15Z |
| 7 | $x^3 + 5x + 5$ | $\{O, (-5, \pm 3), (-4, \pm 3), (-2, \pm 3), (4, \pm 1), (0, \pm 4), (1, 0), (2, \pm 1), (3, \pm 5), (5, \pm 1)\}$ | Z/18Z |
| 8 | $x^3 + 7x + 10$ | $\{O, (-5, \pm 2), (3, \pm 5), (4, \pm 5), (5, \pm 4)\}$ | Z/9Z |
| 9 | $x^3 + 5x + 3$ | $\{O, (-3, \pm 4), (0, \pm 5), (1, \pm 3), (3, \pm 1)\}$ | Z/9Z |
| 10 | $x^3 + 7x$ | $\{O, (-5, \pm 4), (-2, 0), (-1, \pm 5), (0, 0), (2, 0), (3, \pm 2), (4, \pm 2)\}$ | Z/12Z |

| $\alpha$ | $\widetilde{E}_\alpha/\mathrm{F}_{13}$ | Point | Torsion Group |
|---|---|---|---|
| 0 | $x^3$ | $\{O, (-4, \pm 1), (-3, \pm 5), (-1, \pm 5), (0, 0), (1, \pm 1), (3, \pm 1), (4, \pm 5)\}$ | cusp |
| 1 | $x^3 + 10x + 11$ | $\{O, (-2, \pm 3), (-1, 0), (1, \pm 3), (2, 0), (3, \pm 4), (5, \pm 2), (6, \pm 1)\}$ | node |
| 2 | $x^3 + 7x + 4$ | $\{O, (-5, 0), (-4, \pm 4), (-1, \pm 3), (0, \pm 2), (1, \pm 5), (2, 0), (3, 0)\}$ | Z/12Z |
| 3 | $x^3 + 8x$ | $\{O, (-6, \pm 3), (-5, \pm 2), (-3, \pm 1), (5, \pm 3), (-1, \pm 2), (0, 0), (1, \pm 3), (3, \pm 5), (6, \pm 2)\}$ | Z/18Z |
| 4 | $x^3 + 6x + 11$ | $\{O, (-5, \pm 5), (-4, \pm 1), (-2, \pm 2), (-1, \pm 2), (3, \pm 2), (5, \pm 6), (6, \pm 4)\}$ | Z/15Z |
| 5 | $x^3 + 9x + 5$ | $\{O, (-5, \pm 2), (-4, \pm 3), (-3, \pm 4), (4, \pm 1)\}$ | Z/9Z |
| 6 | $x^3 + x + 11$ | $\{O, (-6, \pm 6), (-2, \pm 1), (-1, \pm 3), (1, 0), (4, \pm 1), (6, \pm 5)\}$ | Z/12Z |
| 7 | $x^3 + 7x$ | $\{O, (-5, \pm 3), (-4, \pm 5), (-3, \pm 2), (5, \pm 2), (-2, \pm 2), (0, 0), (2, \pm 3), (3, \pm 3), (4, \pm 1)\}$ | Z/18Z |
| 8 | $x^3 + 2x + 10$ | $\{O, (-6, \pm 4), (-4, \pm 4), (-3, \pm 4), (0, \pm 6), (2, \pm 3), (1, 0), (3, \pm 2), (4, \pm 2), (6, \pm 2)\}$ | Z/18Z |
| 9 | $x^3 + 2x + 8$ | $\{O, (-6, \pm 1), (-5, \pm 4), (-4, \pm 1), (-3, \pm 1), (5, 0), (-2, \pm 3)\}$ | Z/12Z |
| 10 | $x^3 + 12x + 9$ | $\{O, (-4, \pm 1), (-2, \pm 4), (-1, \pm 3), (0, \pm 3), (1, \pm 3), (4, \pm 2), (5, \pm 5)\}$ | Z/15Z |
| 11 | $x^3 + 10x$ | $\{O, (-3, \pm 3), (-1, \pm 3), (0, \pm 6), (4, \pm 3)\}$ | Z/9Z |
| 12 | $x^3 + x + 12$ | $\{O, (-5, \pm 5), (-4, \pm 3), (-1, \pm 6), (0, \pm 5), (1, \pm 1), (2, \pm 3), (3, \pm 4), (5, \pm 5), (6, 0)\}$ | Z/18Z |

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# RANDOM FIXED POINTS FOR $s$-CONDENSING MAPPINGS

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( Received August 31, 2001 )

Submitted by K. K. Azad

## Abstract

Since Bharucha-Reid [Random Integral Equations, Academic Press, New York, London, 1972; Bull. Amer. Math. Soc. 82 (1976), 641-657] proved the stochastic version of the well-known Schauder's fixed point theorem, random fixed point theory found its applications. The study of random fixed point theory has been developed rapidly in recent years, see, e.g., Itoh [J. Math. Anal. Appl. 67(3) (1979), 261-273], Lin [Proc. Amer. Math. Soc. 103 (1988), 1129-1135; Proc. Amer. Math. Soc. 105 (1989), 66-69], Sehgal and Singh [Proc. Amer. Math. Soc. 95 (1985), 91-94], Sehgal and Walters [Contemporary Operators 21 (1983), 215-218; Proc. Amer. Math. Soc. 90 (1984), 425-429], Tan and Yuan [J. Math. Anal. Appl. 185 (1994), 378-390; Stochastic Anal. Appl. 15(1) (1997), 103-123], Yuan and Yu [Nonlinear Anal. TMA 26(6) (1996), 1097-1102], Xu [Proc. Amer. Math. Soc. 110 (1990), 395-400]. The purpose of this paper is to continue discussions of the random fixed point theorems and stochastic version of Ky Fan's best approximation theorem.

## 1. Introduction

Throughout this paper, $(\Omega, \Sigma)$ denotes a complete $\sigma$-finite

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measurable space. Let $X$ denote a complete separable metric space and $F : \Omega \to X$ be a set-valued map with closed images. The map $F$ is called *measurable* (*weakly measurable*) if the inverse image of each closed (open) set is a measurable set. Weakly measurable and measurable maps coincide in a complete σ-finite measure space.

If $(\Omega, \Sigma)$ is a measurable space, then (2) $\Leftrightarrow$ (3) $\Rightarrow$ (1). A measurable mapping $x : \Omega \to X$ is called a *measurable selection* of a measurable set valued map $F : \Omega \to X$ if $x(\omega) \in F(\omega)$ for each $\omega \in \Omega$. Let $M$ be a nonempty closed subset of $X$. Then a mapping $f : \Omega \times M \to X$ is called a *random operator* if, for each fixed $x$ in $M$, the map $f(\cdot, x) : \Omega \to X$ is measurable. A measurable operator $x : \Omega \to X$ is said to be a *random fixed point* of a random operator $f : \Omega \times M \to X$ if $x(\omega) \in M$ and $f(\omega, x(\omega)) = x(\omega)$. Our general reference for set-valued analysis is [2].

In 1967, Sadovskii [13] defined the condensing operators as follows: Let $(X, d)$ be a metric space and $D \subset X$, $D$ nonempty and bounded. A continuous map $f : D \to D$ is called *condensing* if for every $A \subset D$, with $\alpha(A) > 0$, $\alpha(f(A)) < \alpha(A)$, where $\alpha$ is the measure of noncompactness. Daher [7] introduced the condensing type mappings. Let $X$ be a Banach space and $f : X \to X$ be a continuous mapping. Then

(1) for every bounded $A \subset X$, $\alpha(A)$ the measure of noncompactness

$$\alpha(A) = \inf\{\varepsilon > 0 : A \text{ has a finite } \varepsilon - net\}.$$

(2) $\overline{A} = \bigcap\{F \subset X : A \subset F, F \text{ closed}\}$, any $A \subset X$.

(3) $\hat{A} = \bigcap\{C \subset X : A \subset C, C \text{ convex}\}$, any $A \subset X$.

(4) $Kf = \bigcup_{n>=0}\{K_n : K_0 = \hat{K}, K_n = f(K_{n-1}) \subset X\}$ for a nonempty compact subset $K$ of $X$.

**Definition 1** [7]. A continuous mapping $f : X \to X$ is said to be *sequentially condensing* or *s-condensing* if for every bounded $Kf \subset X$ with $\alpha(Kf) > 0$ and $f(Kf)$ bounded, $f$ satisfies the condition $\alpha(f(Kf)) < \alpha(Kf)$.

The retraction map on $X$ is $s$-condensing but not condensing (see [7]). Here we illustrate some examples on $s$-condensing mappings (cites [7]).

**Example 1.** The identity map on $X$ is $s$-condensing but not condensing.

**Example 2.** For each $A \subset X$, convex, non-empty and closed, the retraction $r : X \to A$ is $s$-condensing.

**Example 3.** In $\ell^\infty$, let $r : V \to A$ be a retraction of the closed unit ball centered at the origin, on the set $A = A_0$, where $A_0$ is defined as follows:

$$A_0 = \{\overline{x}_i \in V : \overline{x}_i = (x_1, x_2, ---, x_n, ---),$$

$$x_n = 1, \text{ for all } n \neq i, x_i = -1, i \geq 1\}.$$

That $r(A) = A$ implies $\alpha(A) = 1 = \alpha(r(V))$. Since $\alpha(V) = 1$, it is seen that $r$ is not condensing.

**Remark.** $s$-condensity is not trivially satisfied. Let $V \subset \ell^\infty$ be the unit closed ball. Then $f : V \to V \subset \ell^\infty$ defined by $f((x_1, x_2, x_3, ---)) = (0, x_1, x_2, x_3, ---)$ is an isometry and $\alpha(f(A)) = \alpha(A)$, where $A \subset V$. Take $V^n$ to be any finite dimensional closed ball in $V$ and verify that $V^n f \subset V$, so it is bounded. For any $\overline{x} \in V$, $\overline{x} \neq 0$, the sequence $\langle f^i \overline{x} \rangle \subset V^n f$ has no Cauchy subsequence. Therefore, $V^n f$ is not compact.

**Theorem A** [7]. *Let $C$ be a nonempty closed bounded convex subset of a Banach space $X$ and $f : C \to C$ be s-condensing. Then $f$ has a fixed point.*

**Theorem B** [7]. *If $f : C \to X$ is s-condensing and $R : X \to C$ is a retraction, then $R \circ f : C \to C$ is s-condensing.*

## 2. Main Results

**Theorem 2.1.** *Let $(\Omega, \Sigma)$ be a complete σ-finite measurable space and*

*C be a nonempty bounded closed convex subset of a separable Banach space X. If $f : \Omega \times C \to C$ is a random s-condensing operator, then f has a random fixed point in C.*

**Proof.** Define a multivalued map $F : \Omega \to 2^C \setminus \phi$ by $F(\omega) = \{x \in C : f(\omega, x) = x\}$. Then $F$ is nonempty closed valued because, for each $\omega$, $f$ has a fixed point from Theorem A. Consider the graph of $F$,

$$Gr(F) = \{(\omega, x) \in \Omega \times C : f(\omega, x) = x\}$$

$$= \{(\omega, x) \in \Omega \times C : \| f(\omega, x) - x \| = 0\} \in \sum \otimes \mathcal{B}(X),$$

where $\mathcal{B}(X)$ is the Borel $\sigma$-field of $X$. Since the functional $\| f(\omega, x) - x \|$ on $\Omega \times C$ is measurable with respect to $\omega$ and continuous with respect to $x$, respectively, $F$ is measurable. The measurable selection theorem (see, [2]) on a complete $\sigma$-finite measurable space guarantees that there exists a measurable selection $x : \Omega \to C$ such that $x(\omega) \in F(\omega)$ for each $\omega \in \Omega$. Therefore, $x(\omega) = f(\omega, x(\omega))$ for each $\omega \in \Omega$.

From Assad's result [1] and Theorem 2.1, we can obtain the following as the direct consequence.

**Theorem 2.2.** *Let C be a separable, closed and convex subset of a reflexive Banach space X and $f : \Omega \times C \to X$ be s-condensing random operator. Suppose that for any $\omega \in \Omega$,*

(i) *$f(\omega, C)$ is bounded*

(ii) *$f(\Omega \times \partial C) \subset C$.*

*Then f has a random fixed point.*

**Remark.** It would be interesting to know whether Theorem 2.1 remains true under the assumption that $(\Omega, \Sigma)$ is a complete $\sigma$-algebra. A condensing random fixed point theory on a complete $\sigma$-algebra has been studied by K. K. Tan and X. Z. Yuan [19], X. Z. Yuan and J. Yu [21], and H. K. Xu [20]. Especially, the hemicompactness is helpful to prove the random fixed point of condensing map. However, the $s$-condensing map is not hemicompact, in general. The retraction map is not hemicompact. In

what follows, we shall prove a random best approximation theorem for random continuous $s$-condensing mapping defined on balls of separable Banach spaces. The deterministic forms of theorems were studied by K. Fan [8] for continuous maps and by S. P. Singh and B. Watson [17] for $s$-condensing mapping.

**Theorem 2.3.** *Let $\overline{B}_r$ be a closed ball about the origin of radius $r > 0$ in a separable Banach space $X$, and $f : \Omega \times \overline{B}_r \to X$ be $s$-condensing random operator. Then there exists a measurable map $\phi : \Omega \to \overline{B}_r$ such that $\| \phi(\omega) - f(\omega, \phi(\omega)) \| = d(f(\omega, \phi(\omega)), \overline{B}_r)$ for each $\omega \in \Omega$.*

**Proof.** Let $R$ be a retraction of $X$ onto $\overline{B}_r$ given by

$$R(x) = \begin{cases} x & \text{if } \| x \| \leq r \\ \dfrac{rx}{\| x \|} & \text{if } \| x \| \geq r. \end{cases}$$

Define $F : \Omega \times \overline{B}_r \to \overline{B}_r$ by $F(\omega, x) = R \circ f(\omega, x)$. Then, by Theorem B, $F$ is $s$-condensing random operator. From Theorem 2.1, $F$ has a random fixed point, i.e., there exists a measurable map $\rho : \Omega \to \overline{B}_r$ such that $\rho(\omega) = F(\omega, \rho(\omega))$ for each $\omega \in \Omega$. If we follow the lines of the proof [18, 19], we have

$$\| \rho(\omega) - f(\omega, \rho(\omega)) \| = d(f(\omega, \rho(\omega)), \overline{B}_r).$$

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# AN ARITHMETIC PROOF OF JACOBI'S EIGHT SQUARES THEOREM

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Submitted by K. K. Azad

**Abstract**

An elementary proof of Jacobi's eight squares theorem is given.

## 0. Notation

Let $n$ and $s$ denote positive integers. We let $r_s(n)$ denote the number of representations of $n$ as the sum of $s$ squares. We also let

$$\sigma_s(n) = \sum_{d \mid n} d^s, \qquad \sigma(n) = \sigma_1(n),$$

where $d$ runs through the positive integers dividing $n$. If $x$ is not a positive integer, we set $\sigma_s(x) = 0$. We also define

$$A_s(n) = \sum_{k<n/s} \sigma(k)\,\sigma(n - sk),$$

where the summation is over all integers $k$ satisfying $1 \le k < n/s$. Finally, we set

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$$F_s(n) = \begin{cases} 1, & \text{if } s \mid n, \\ 0, & \text{if } s \nmid n. \end{cases}$$

## 1. Introduction

The formula

$$r_8(n) = 16(-1)^n \sum_{d \mid n} (-1)^d d^3 \tag{1}$$

first appeared implicitly in the work of Jacobi [5], [6, Sections 40-42] and explicitly in the work of Eisenstein [2], [3, p. 501]. The standard arithmetic proof of (1) uses an elementary identity due to Liouville [8], see [10, p. 402], to show that the function on the right hand side of (1) satisfies the same recurrence relation as $r_8(n)$ with the same initial conditions so that the two functions are the same, see, for example, [10, pp. 441-445]. It is the purpose of this note to give a different arithmetic proof of (1). Our starting point is the following elementary identity due to Huard, Ou, Spearman and Williams [4], which is an extension of an identity of Liouville [7, p. 284].

**Huard-Ou-Spearman-Williams Identity.** Let $f : \mathbb{Z}^4 \to \mathbb{C}$ be such that

$$f(a, b, x, y) - f(x, y, a, b) = f(-a, -b, x, y) - f(x, y, -a, -b)$$

for all integers $a$, $b$, $x$ and $y$. Then

$$\begin{aligned} &\sum_{ax+by=n} (f(a, b, x, -y) - f(a, -b, x, y) + f(a, a-b, x+y, y) \\ &-f(a, a+b, y-x, y) + f(b-a, b, x, x+y) - f(a+b, b, x, x-y)) \\ &= \sum_{d \mid n} \sum_{x<d} (f(0, n/d, x, d) + f(n/d, 0, d, x) + f(n/d, n/d, d-x, -x) \\ &-f(x, x-d, n/d, n/d) - f(x, d, 0, n/d) - f(d, x, n/d, 0)), \end{aligned} \tag{2}$$

where the sum on the left hand side of (2) is over all positive integers $a$, $b$, $x$, $y$ satisfying $ax + by = n$, the inner sum on the right hand side is

over all positive integers $x$ satisfying $x < d$, and the outer sum on the right hand side is over all positive integers $d$ dividing $n$.

The proof in [4, Section 2] of this identity is completely elementary as it only involves the rearrangement of terms in finite sums. The choice $f(a, b, x, y) = xy$ in (2) yields the identity [4, eqn. (16)]

$$A_1(n) = \frac{1}{12}(5\sigma_3(n) + (1 - 6n)\sigma(n)), \tag{3}$$

which originally appeared in a letter from Besge to Liouville [1]. The choice $f(a, b, x, y) = (2a^2 - b^2) F_4(x)$ yields the identity [4, Theorem 4]

$$A_4(n) = \frac{1}{48}(\sigma_3(n) + 3\sigma_3(n/2) + 16\sigma_3(n/4)$$

$$+ (2 - 3n)\sigma(n) + (2 - 12n)\sigma(n/4)), \tag{4}$$

which is an extension of a result of Melfi [9, eqn. (11)]. The choice $f(a, b, x, y) = \left(\frac{-4}{ab}\right)$ (Legendre-Jacobi-Kronecker symbol) gives Jacobi's four squares formula [4, Section 7]

$$r_4(n) = 8\sigma(n) - 32\sigma(n/4). \tag{5}$$

Another arithmetic proof of (5) has been given by Spearman and Williams [11]. Thus formulae (3), (4), (5) can all be proved by entirely elementary means. We now use these three results to give an arithmetic proof of (1).

## 2. Arithmetic Proof of Jacobi's Eight Squares Theorem

We have

$$r_8(n) = \sum_{k=0}^{n} r_4(k) r_4(n-k) = 2r_4(n) + \sum_{k=1}^{n-1} r_4(k) r_4(n-k), \tag{6}$$

as $r_4(0) = 1$. Appealing to (5), we obtain

$$\sum_{k=1}^{n-1} r_4(k) r_4(n-k) = 64S_1 - 256S_2 - 256S_3 + 1024S_4, \tag{7}$$

where

$$S_1 = \sum_{k=1}^{n-1} \sigma(k)\,\sigma(n-k), \tag{8}$$

$$S_2 = \sum_{k=1}^{n-1} \sigma(k/4)\,\sigma(n-k), \tag{9}$$

$$S_3 = \sum_{k=1}^{n-1} \sigma(k)\,\sigma((n-k)/4), \tag{10}$$

$$S_4 = \sum_{k=1}^{n-1} \sigma(k/4)\,\sigma((n-k)/4). \tag{11}$$

Clearly $S_1 = A_1(n)$ and changing the summation variable in (10) from $k$ to $n-k$ shows that $S_3 = S_2$. Since the only terms in $S_2$ and $S_4$ which do not vanish are those for which $4 \mid k$, replacing $k$ by $4k$ in (9) and (11), we find that $S_2 = A_4(n)$ and $S_4 = A_1(n/4)$. Appealing to (3) and (4) for the values of $A_1(n)$ and $A_4(n)$, and to (5) for the value of $r_4(n)$, we obtain from (6)-(11)

$$r_8(n) = 16\sigma_3(n) - 32\sigma_3(n/2) + 256\sigma_3(n/4). \tag{12}$$

Examining the three possibilities $2 \nmid n$, $2 \parallel n$ and $4 \mid n$ individually, we find that the right hand side of (12) is the same as the right hand side of (1).

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# NULL FRÉNET'S FRAMES AND NULL FRÉNET'S CURVES IN LORENTZIAN SPACES OF ZERO CURVATURE

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Submitted by K. K. Azad

## Abstract

In 1969, Bonnor [Tensor (N.S.) 20 (1969), 229-242] showed that in 4-dimensional Lorentz space any null curve has 3 curvatures. In this paper we show the number of linear independent curvatures and characterize congruent null curves in $n$-dimensional Lorentz spaces.

## 1. Introduction

Frénet's frames and Frénet's curves for $m$-dimensional Riemannian manifolds of constant curvature were studied by Muñoz Masqué and Rodríguez Sánchez in [5]. In that paper the authors showed the corresponding Frénet formulas in terms of $(m-1)$ curvatures.

For Lorentz-Minkowski manifolds $L^3$ and $L^4$ of zero curvature we found different moving frames attached to null curves: Frénet equations [1], Cartan tetrad [1], [2], [3], and Frénet equations for null frames [3].

From [1] and [3], it is easy to see the natural generalization of Cartan tetrad from $L^3$ to $L^4$, depending on 2 and 3 curvatures, respectively. But

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from [1] we have that Frénet equations from general frames in $L^4$ depend on 4 curvatures.

The aim of this paper is to generalize the Frénet frames for null curves and obtain the curvatures for these curves in the Lorentzian space $L^n$. We prove that in $L^n$ the null curves have $\frac{1}{2}\{n^2 - 3n + 6\}$ linear independent curvatures.

We also characterize congruent curves in $L^n$.

## 2. Preliminaries

Let $(L^n, g)$ be an $n$-dimensional Lorentzian space of zero curvature where the signature of $g$ is $(+, -, \cdots, -)$ and $\nabla$ its Levi-Civita connection. We will indicate with dot the corresponding inner product.

It is well known that in Lorentzian spaces there are three kinds of vectors: timelike, spacelike and null, according to the inner product with itself be $-1$, 1 or 0, respectively.

In what follows we deal with curves for which the tangent vector at every point is a null one.

**Definition 1.** A curve $\sigma : (a, b) \to L^n$ is said to be a *null curve* if, its tangent vector $T = \sigma_*\left(\frac{d}{dt}\right)$ is (nonzero) null at every point of $\sigma(s)$, i.e., $T \cdot T = 0$.

From now on we will assume that the curves are parametrized by proper time parameter $t$, which is the Lorentzian counterpart of the arc length parameter.

Following the Muñoz Masqué and Rodríguez Sánchez's idea, we extend the characterization of Frénet curves.

## 3. Null Frénet Frames and Curves

**Definition 2.** A curve $\sigma : (a, b) \to L^n$ is said to be a *null Frénet*

*curve* at the point $t_0 \in (a, b)$ if, the tangent vectors to $\sigma(t)$ at $t_0$, $T_{t_0}, (\nabla_T T)_{t_0}, \ldots, (\nabla_T^{n-1} T)_{t_0}$ are linear independent and $g(T, T) = 0$.

**Definition 3.** Let $\sigma$ be a null curve in $L^n$ and $P$ be a point of the curve $\sigma$. A *null frame* at $P$ consists of a set of vectors $\{X_1, \ldots, X_n\}$ which satisfy the following conditions:

$$X_1 = k_0 T = k_0 \sigma_*\left(\frac{d}{dt}\right),\ k_0 > 0,\quad X_1 \cdot X_1 = 0,\quad X_2 \cdot X_2 = 0,\quad X_1 \cdot X_2 = -1$$

$$X_i \text{ spacelike vectors}, \qquad X_1 \cdot X_i = X_2 \cdot X_i = 0 \qquad i : 3, \ldots, n$$

$$X_i \cdot X_j = \delta_{ij} \qquad\qquad i, j : 3, \ldots, n.$$

**Definition 4.** If $\sigma$ is a null Frénet curve in $L^n$ at the point corresponding to $t \in (a, b)$, we can define a *null Frénet frame* of $\sigma$ impossing the following two conditions:

1. $\{T_t, (\nabla_T T)_t, \ldots, (\nabla_T^{n-1} T)_t\}$ and $\{X_1, \ldots, X_n\}$ span the same subspace.

2. $\det_{\{X_1, \ldots, X_n\}}(T_t, (\nabla_T T)_t, \ldots, (\nabla_T^{n-1} T)_t) > 0$.

As $\{X_1, \ldots, X_n\}$ spans the tangent space at $P$, we have

$$T = k_0 X_1 \qquad X_i' = \sum_j w_{ij} X_j, \tag{1}$$

where $w_{ij}$ are continuously differentiable functions of $t$.

We require that if the conditions of Definition 3 have to be preserved along the curve, then $\frac{d}{dt}(X_i \cdot X_j) = 0$ and applying these to (1), we obtain

$$\left.\begin{array}{ll} w_{12} = w_{21} = 0 & \\ w_{i1} = w_{2i} & i : 3, \ldots, n \\ w_{i2} = w_{1i} & i : 3, \ldots, n \\ w_{ij} = -w_{ji} & i, j : 3, \ldots, n \end{array}\right\}. \tag{2}$$

From [1] we introduced a generalized version of the concept of null rotation.

**Definition 5.** A *null rotation* is a Lorentzian transform which preserves orientation and null frames.

**Theorem 6.** *Let* $\sigma(s)$ *be a null curve in* $L^n$, $n \geq 4$, *and* $\{X_1, \ldots, X_n\}$ *be a null frame attached at every point of* $\sigma(s)$. *Then* $\sigma(s)$ *has* $\frac{1}{2}(n^2 - 3n + 6)$ *linear independent curvatures.*

**Proof.** We want to prove it by induction on the dimension of the space.

Let $P$ be a point of $\sigma(s)$ and $T$ be the tangent vector to $\sigma(s)$ at $P$. Without loss of generality, we can take $T = X_1$, $X_i' = \sum_j w_{ij} X_j$ for $i, j : 1, \ldots, n$, under the relations $w_{12} = w_{21} = w_{1n} = w_{n2} = 0$, $w_{i1} = w_{2i}$ and $w_{i2} = w_{1i}$, $i : 3, \ldots, n$, $w_{ij} = -w_{ji}$, $i, j : 3, \ldots, n$.

We assume that for $L^{n-1}$, we have $\frac{1}{2}(n^2 - 5n + 10)$ linear independent curvatures.

From $L^{n-1}$ to $L^n$ we add $n + (n-1)$ new curvatures but $n$ of them are dependent and we miss one more for the null rotation $(w_{1n} = w_{(n-1)1})$.

Then, for $L^n$, we have $\frac{1}{2}(n^2 - 3n + 4) + n + (n-1) - n - 1 = \frac{1}{2}(n^2 - 3n + 4) + n - 2 = \frac{1}{2}(n^2 - 3n + 6)$ different curvatures.

In particular, for $n = 4$ we get

$$X_1' = w_{13} X_3$$

$$X_2' = w_{23} X_3 + w_{24} X_4$$

$$X_3' = w_{23} X_1 + w_{13} X_2 + w_{34} X_4$$

$$X_4' = w_{24} X_1 - w_{34} X_3.$$

**Corollary 7.** *Denoting the $k_i$ curvature by $k_i(L^j)$ in $L^j$, we have for* $i : 1, \ldots, \frac{1}{2}\{(j^2 - 3j + 4)\},\ k_i(L^j) = k_i(L^{j+1})$.

**Proof.** Denoting by $k_i(s), i : 1, \ldots, \frac{1}{2}(n^2 - 3n + 6)$ the obtained curvatures we write the following matrix associated to this system:

$$\begin{pmatrix} 0 & 0 & k_2 & k_5 & * & k_{\frac{1}{2}\{n^2-3n+6\}} \\ 0 & 0 & k_1 & k_3 & * & k_{\frac{1}{2}\{n^2-5n+12\}} \\ k_1 & k_2 & .0 & k_4 & * & k_{\frac{1}{2}\{n^2-5n+14\}} \\ k_5 & k_5 & -k_4 & 0 & * & * \\ * & * & * & * & 0 & k_{\frac{1}{2}\{n^2-3n+6\}} \\ -k_{\frac{1}{2}\{n^2-5n+12\}} & k_{\frac{1}{2}\{n^2-3n+6\}} & -k_{\frac{1}{2}\{n^2-5n+14\}} & * & * & 0 \end{pmatrix}$$

$$= (a_{ij})_{i,\ j:1, \ldots, n} = A.$$

We observe that the $3 \times 3$ block of $A$, $(a_{ij})$, $i, j : 1, 2, 3$, corresponds to null Frénet formulas for a null curve in $L^3$, the $4 \times 4$ block of $A$, $(a_{ij})$, $i, j : 1, 2, 3, 4$, corresponds to the null Frénet formulas for a null curve in $L^4$ and the curvatures which appear in the 4th column and row are independent of those of $L^3$. The whole matrix has this characteristic. From all above and denoting $k_i(L^j)$ the $k_i$ curvature in $L^j$ we have for $i : 1, \ldots, \frac{1}{2}\{(j^2 - 3j + 6)\},\ k_i(L^j) = k_i(L^{j+1})$.

**Corollary 8.** *The statement of Theorem 6 holds for $n \geq 3$.*

**Proof.** Originally [1], for $n = 3$ Bonnor showed the corresponding matrix $\begin{pmatrix} k & 0 & k_2 \\ 0 & -k & k_1 \\ k_1 & k_2 & 0 \end{pmatrix}$ and applying a null rotation obtained $k = 0$.

**Remark 9.** $X_1'$ and $X_2'$ are always spacelike vectors.

## 4. Congruent Null Curves in $L^n$

According to [4] and [5] we can characterize two congruent curves in a simply connected complete Riemannian manifolds of constant curvature. As $L^n$ is simply connected complete semi-Riemannian manifolds of constant curvature and, [6], the Levi-Civita connection $\nabla$ preserves all its characteristics in a semi-Riemannian manifold, we state

**Definition 10.** Two null curves $\sigma_1$ and $\sigma_2$ in $L^n$ are said to be *congruents* if there exists an open isometric embedding $\Phi$ such that $\sigma_2 = \Phi(\sigma_1)$.

In order to characterize congruent null curves in $L^n$ in terms of its curvatures appeared in the null Frénet frame, we have

**Theorem 11.** *Let $\sigma_1$ and $\sigma_2 : (a, b) \to L^n$ be two null Frénet curves with values in $L^n$ and $T_1$ and $T_2$ be their tangent vectors, respectively, for any $t \in (a, b)$ and $k_i^1(t)$ and $k_i^2(t)$ the corresponding curvatures. Then*

(i) *$\sigma_1$ and $\sigma_2$ are congruent curves in a neighbourhood of $t_0$ if and only if $k_i^1(t) = k_i^2(t)$ for $i = 0, 1, \ldots, n$ and for sufficiently small $|t - t_0|$.*

(ii) *$\sigma_1$ and $\sigma_2$ are congruent curves in a neighbourhood of $t_0 \in (a, b)$ if and only if for any $i, j \in N$,*

$$g\left(\left(\nabla^i_{T_1} T_1\right)_{t_0}, \left(\nabla^j_{T_1} T_1\right)_{t_0}\right) = g\left(\left(\nabla^i_{T_2} T_2\right)_{t_0}, \left(\nabla^j_{T_2} T_2\right)_{t_0}\right).$$

**Proof.** Taking into account the topological properties of $L^n$, the signature of $g$ and that the index $i$ runs from 0 to $n-1$ and $\alpha$ runs from 0 to $n$, we can follow step by step Theorem 1 of [5].

**Corollary 12.** *Let $\sigma_1$ be a null curve in $L^q$ and $\sigma_2$ be a null curve in $L^p$, $q < p$. They are congruent in a neighbourhood of $t_0$ if and only if*

$k_i^1(t) = k_i^2(t)$ *for* $i : 1, \ldots, \frac{1}{2}\{q^2 - 3q + 6\}$ *and for sufficiently small* $|t - t_0|$.

**Proof.** Follows applying Corollary 7 of Theorem 6.

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# REMARK ON RATIONAL APPROXIMATIONS TO $e^{1/k}$

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Submitted by K. K. Azad


## Abstract

For any $\varepsilon$ with $0 < \varepsilon < \frac{1}{2}$, there exists an effectively computable integer $N^* = N^*(\varepsilon)$ such that

$$\left| e - \frac{p}{q} \right| > \left( \frac{1}{2} - \varepsilon \right) \frac{\log \log q}{q^2 \log q}$$

for all integers $p$, $q$ with $q \geq N^*$. Corresponding results are given for the inequality if $e$ be replaced by $e^{1/2}, e^{1/3}, e^{1/4}, \ldots$.


## 1. Introduction

Davis [1] proved the following theorem:

*Let $k$ be a positive integer. Then, for any $\varepsilon > 0$, there exists a number $q' = q'(k, \varepsilon)$ such that*

$$\left| e^{1/k} - \frac{p}{q} \right| > \left( \frac{1}{2k} - \varepsilon \right) \frac{\log \log q}{q^2 \log q}$$

*for all integers $p$, $q$ with $q \geq q'$.*

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Davis' result can be improved as follows:

**Theorem 1.** *For any* $\varepsilon$ *with* $0 < \varepsilon < \frac{1}{2}$, *let* $N$ *be an integer such that*

$$N \geq 3\left(\frac{1}{\left(\frac{1}{2}-\varepsilon\right)(1+\varepsilon)} - 2\right)^{-1} - \frac{1}{2}, \qquad 4N + 7 \leq e^{(N+7/4)^{\varepsilon}+1} \ (N \geq 6).$$

*Then*

$$\left| e - \frac{p}{q} \right| > \left(\frac{1}{2} - \varepsilon\right) \frac{\log\log q}{q^2 \log q}$$

*for all integers* $p, q$ *with* $q \geq \prod_{m=1}^{N}(4m+3)$.

**Example.** For $\varepsilon = 2^{-10}$, we can define that $N = 2^{2^{14}} - 1$.

**Theorem 2.** *Let* $k \geq 2$ *be a positive integer. For any* $\varepsilon$ *with* $0 < \varepsilon < \frac{1}{2k}$, *let* $N$ *be an integer such that*

$$N \geq (2k+1)\left(\frac{1}{\left(\frac{1}{2k}-\varepsilon\right)(1+\varepsilon)} - 2k\right)^{-1} + \frac{1}{2}, \ 4k(N+1) \leq e^{(N+1)^{\varepsilon}+1} \ (N \geq 5).$$

*Then*

$$\left| e^{1/k} - \frac{p}{q} \right| > \left(\frac{1}{2k} - \varepsilon\right) \frac{\log\log q}{q^2 \log q}$$

*for all integers* $p, q$ *with* $q \geq (4k)^N N!$.

**Example.** For $2 \leq k \leq 8$ and $\varepsilon = 2^{-10}$, we can define that $N = 2^{2^{14}} - 1$.

## 2. Lemmas

**Lemma 1** (cf. [2] Lemma). *Let* $p_n/q_n$ *be the* $n$*-th convergent of* $e$.

*Then*

$$\left| e - \frac{p}{q} \right| > \frac{\log \log q}{\gamma_N q^2 \log q}$$

*for all integers* $p$, $q$ *with* $q \geq q_{3N+1}$ $(N \geq 6)$, *where*

$$\gamma_N = \left(2 + \frac{3}{N + 1/2}\right)\left(1 + \frac{\log\log((4N + 7)/e)}{\log(N + 7/4)}\right).$$

**Lemma 2** (cf. [3] Lemma). *Let* $k \geq 2$ *be a positive integer, and let* $p_n/q_n$ *be the n-th convergent of* $e^{1/k}$. *Then*

$$\left| e^{1/k} - \frac{p}{q} \right| > \frac{\log \log q}{\delta_N q^2 \log q}$$

*for all integers* $p$, $q$ *with* $q \geq q_{3N}$ $(N \geq 5)$, *where*

$$\delta_N = 2\left(k + \frac{2k + 1}{2N - 1}\right)\left(1 + \frac{\log\log(4k(N + 1)/e)}{\log(N + 1)}\right).$$

## 3. Proofs of Theorems

**I. Proof of Theorem 1.** Let $p_n/q_n$ be the $n$-th convergent of $e$. The continued fraction of $e$ is

$$e = [a_0, a_1, a_2, a_3, \cdots] = [2, \overline{1, 2n, 1}]_{n=1}^{\infty}.$$

In other words, $a_0 = 2$, and for $m \geq 1$, $a_{3m} = a_{3m-2} = 1$ and $a_{3m-1} = 2m$.

Since

$$\begin{aligned} q_{3N+1} &= a_{3N+1} q_{3N} + q_{3N-1} = q_{3N} + q_{3N-1} = 2q_{3N-1} + q_{3N-2} \\ &= (4N + 1) q_{3N-2} + 2q_{3N-3} \leq (4N + 3) q_{3N-2}, \end{aligned}$$

we have

$$q_{3N+1} \leq \prod_{m=1}^{N} (4m + 3).$$

Since

$$N \geq 3\left(\frac{1}{\left(\frac{1}{2}-\varepsilon\right)(1+\varepsilon)} - 2\right)^{-1} - \frac{1}{2}, \qquad 0 < \frac{\log\log((4N+7)/e)}{\log(N+7/4)} \leq \varepsilon$$

and $\dfrac{\log\log((4N+7)/e)}{\log(N+7/4)}$ $(N \geq 6)$ is a strictly decreasing function, we obtain

$$\frac{1}{2} - \varepsilon \leq \frac{1}{2+\dfrac{3}{N+1/2}} \cdot \frac{1}{1+\varepsilon} \leq \frac{1}{2+\dfrac{3}{N+1/2}} \cdot \frac{1}{1+\dfrac{\log\log((4N+7)/e)}{\log(N+7/4)}}.$$

And from Lemma 1, for all positive integers $p, q$ with $q \geq \prod_{m=1}^{N}(4m+3)$,

$$\left|e - \frac{p}{q}\right| > \frac{\log\log q}{\gamma_N q^2 \log q} \geq \left(\frac{1}{2} - \varepsilon\right)\frac{\log\log q}{q^2 \log q}.$$

This completes the proof.

**II. Proof of Theorem 2.** We can prove Theorem 2 by Lemma 2. The proof is completely analogous to the previous proof.

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# THE SCHUR MULTIPLICATOR OF TOPOLOGICAL GROUPS; RELATIONS AMONG COMMUTATORS

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Submitted by K. K. Azad


## Abstract

Let $X$ be a completely regular Hausdorff topological space. By $F(X)$ we mean the free topological group on $X$ in the Markov sense [Amer. Math. Soc. Transl. 30 (1950), 11-88]. In this paper, we assign a topological group $H(G)$ to a topological group $G$ and show that there is a continuous homomorphism and one-one correspondence between $H(G)$ and the Schur multiplicator of topological groups [Kyungpook Math. J. 31(1) (1991), 35-71]. The method involves using the relations among commutators of $G$ and the short exact sequence

$$R \to F \to G,$$

where $F$ is the free topological group on $G$.


## 1. Introduction

Throughout, $G$ is a topological group and $F(G)$ is the *free topological group* on a completely regular space in Markov sense [3].

By a free topological presentation of $G$ we mean a short exact sequence,

$$R \xrightarrow{i} F \xrightarrow{\pi} G,$$

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with a continuous section $u : G \hookrightarrow F$, i.e., $\pi u = \mathrm{Id}$. If $H$ is a normal subgroup, then $\overline{H}$ is normal [6].

Let $X$ be a completely regular space. We call $F(X)$ to be a *free topological group* on $X$ if

(1) $X$ is a subspace of $F(X)$.

(2) $X$ generates $F(X)$ topologically.

(3) Every continuous function $\phi : X \to H$, where $H$ is any topological group, can be extended to a continuous homomorphism, $\bar{f} : F(X) \to H$, such that $\overline{F}|_X = \phi$.

## 2. The Group $H(G)$

Let $G$ be a topological group and $\langle G, G\rangle$ be the free topological group on $G \times G$, where elements are denoted by $\langle x, y\rangle$, $x, y \in G$. Then there is a natural continuous homomorphism from $\langle G, G\rangle$ to $[G, G]$, the commutator subgroup of $G$, and also a continuous homomorphism $\phi : \langle G, G\rangle \to [G, G]$, $\phi\langle x, y\rangle = [x, y]$.

Let $w \in \langle G, G\rangle$, and $\phi(w) = [w]$, $Z(G) = \{w \in \langle G, G\rangle \mid [w] = 1\}$. Suppose $B(G)$ is the closure of the normal subgroup of $\langle G, G\rangle$ generated by

(1) $\langle x, x\rangle \sim 1$.

(2) $\langle x, y\rangle \sim \langle y, x\rangle^{-1}$.

(3) $\langle xy, z\rangle \sim \langle y, z\rangle^{x}\langle x, z\rangle$.

(4) $\langle y, z\rangle^{x} \sim \langle x, [y, z]\rangle\langle y, z\rangle$, for every $x \in G$, $y \in G$, $z \in G$.

We can conclude that

(5) $\langle x, y\rangle^{x} = \langle y^{x}, z^{x}\rangle = \langle xyx^{-1}, xzx^{-1}\rangle$.

In fact, $B(G)$ is a normal subgroup generated by $\langle x, x\rangle, \langle x, y\rangle\langle y, x\rangle, \dots$. Note that by "~" we mean the congruency in $\langle G, G\rangle \bmod B(G)$. It is clear that $B(G) \subseteq Z(G)$. Now, define

$$H(G) = Z(G)/B(G).$$

If $h : G \to G'$ is a continuous homomorphism of topological groups, then the induced map

$$h_{\#} : \langle G, G\rangle \to \langle G', G'\rangle$$

is given by

$$h_{\#} : \langle x, y\rangle = \langle h(x), h(y)\rangle, \langle x, y\rangle \in \langle G, G\rangle.$$

Now $h_{\#} : (B(G)) \subset B(G')$, $h_{\#}(Z(G)) \subset Z(G')$. The induced map $h_{\#}$ is $h_* : H(G) \to H(G')$. The following hold:

$$(hg)_* = h_* g_*, \quad O_* = O, \quad 1_* = 1,$$

where $O$ is the zero and 1 the identity homomorphism, respectively. Taking inverse of (3) and using (2), we have

(3') $\langle x, yz\rangle \sim \langle x, y\rangle^{y}$.

Among the relations from the definition of $B(G)$, we need the following:

(6) $\langle x, y\rangle^{\langle a, b\rangle} \sim \langle x, y\rangle^{[a, b]}$.

With the definition:

$$\langle x, y\rangle^{\langle a, b\rangle} \sim \langle a, b\rangle\langle x, y\rangle\langle a, b\rangle^{-1},$$

we have

(7) $[\langle x, y\rangle, \langle a, b\rangle] \sim \langle [x, y], [a, b]\rangle$.

(8) $\langle b, b'\rangle\langle a_0, b_0\rangle \sim \langle [b, b'], a_0\rangle\langle a_0, [b, b']b_0\rangle\langle b, b'\rangle$.

(9) $\langle b, b'\rangle\langle b_0, a_0\rangle \sim \langle [b, b']b_0, a_0\rangle\langle a_0, [b, b']\rangle\langle b, b'\rangle$.

(10) $\langle b, b' \rangle \langle a, a' \rangle \sim \langle [b, b'], [a, a'] \rangle \langle a, a' \rangle \langle b, b' \rangle$.

(11) $\langle x^n, x^s \rangle \sim 1$, $n \in Z$, $s \in Z$.

For proving (6) we expand $\langle ax, by \rangle$ by using (3′) and (3):

$$\begin{aligned} \langle ax, by \rangle &\sim \langle ax, b \rangle \langle ax, y \rangle^b \\ &\sim \langle x, b \rangle^a \langle a, b \rangle \langle x, y \rangle^{ba} \langle a, y \rangle^b \langle ax, by \rangle \\ &\sim \langle x, by \rangle^a \langle a, by \rangle \\ &\sim \langle x, b \rangle^{ab} \langle x, y \rangle^{ab} \langle a, b \rangle \langle a, y \rangle^b. \end{aligned}$$

Now, we conclude

$$\langle a, b \rangle \langle x, y \rangle^{ba} \sim \langle x, y \rangle^{ab} \langle a, b \rangle$$

$$\langle a, b \rangle \langle x, y \rangle^{ba} \langle a, b \rangle^{-1} \sim \langle x, y \rangle^{ab}.$$

Changing $x$ to $x^{(ba)^{-1}}$ and $y$ to $y^{(ba)^{-1}}$, we have

$$\langle a, b \rangle \langle x, y \rangle^{ba} \langle a, b \rangle^{-1} \sim \langle x, y \rangle^{[a, b]}.$$

Note that (7) is a consequence of (6) and (4) because:

$$\begin{aligned} [\langle x, y \rangle, \langle a, b \rangle] &= \langle x, y \rangle^{\langle a, b \rangle} \langle x, y \rangle^{-1} \\ &\sim \langle x, y \rangle^{[a, b]} \langle x, y \rangle^{-1} \\ &\sim \langle [a, b], [x, y] \rangle \langle x, y \rangle \langle x, y \rangle^{-1}. \end{aligned}$$

Formula (8) is proved by extending $\langle a_0, [a, b] b_0 \rangle$ using (3′) and (6):

$$\begin{aligned} \langle a_0, [a, b'] b_0 \rangle &\sim \langle a_0, [b, b'] \rangle \langle a_0, b_0 \rangle^{[b, b']} \\ &\sim \langle a_0, [b, b'] \rangle \langle a_0, b_0 \rangle^{\langle b, b' \rangle} \end{aligned}$$

and replacing it in the right side of (8). Formula (9) is the same as (7) and (11). It can be proved by induction on $n + s$, for $n, s \geq 0$, and (3), (3′). If $n + s = 1$, say $n = 0$, $s = 1$, then with $x = 2$ and $y = 1$ in (3) the result is obtained. The general form follows from non-negative case and (3).

**Theorem 2.1.** *If $F$ is a free topological group, then $H(F)$ is a singleton.*

It is enough to prove the theorem for a free topological group with finite number of generators; for if $F$ has infinitely many generators and $G$ is a separable space, then $F(G)$ is separable and by [5] it can be embedded, as a topological subgroup, in a group with two generators. If $F$ has one generator, then by (11), $H(F) = 1$. The case $F$ with finite number of generators follows from the next lemma. The free product of topological groups is as in [4].

**Lemma 2.2.** *If $G = A * B$, then $H(G) \simeq H(A) * H(B)$.*

**Proof.** Let $i : A \to G$, $j : B \to G$ be the natural one to one maps and $p : G \to A$, $q : G \to B$ be the projections. The following diagram shows that $H(B) \simeq j_*H(B)$, $H(A) \simeq i_*H(A)$ and $i_*H(B)$, $j_*H(A)$ are disjoint

$$\begin{array}{c} H(A) \xrightarrow{i_*} H(G) \xleftarrow{j_*} H(B) \\ || \\ H(A) \xleftarrow{p_*} H(G) \xrightarrow{q_*} H(B) \end{array}$$

If $H(G)$ is the product of $i_*H(A)$ and $j_*H(B)$, then $H(G) = j_*H(A) \times i_*H(B)$. We show that $H(G) = i_*H(A)\, j_*H(B)$.

Consider three subgroups of $\langle G, G\rangle$ namely: $\mathcal{A} = i_\#\langle A, A\rangle$, $\mathcal{B} = j_\#\langle B, B\rangle$, $\mathcal{M}$ is the subgroup generated by $\langle a, b\rangle$ such that $1 \neq a \in A$, $1 \neq b \in B$. Let $\langle x, y\rangle$ be a generator of $\langle G, G\rangle$ such that

$$x = a_1 b_1 a_2 b_2 \dots a_k b_k,$$

$$y = \overline{a}_1 \overline{b}_1 \overline{a}_2 \overline{b}_2 \dots \overline{a}_k \overline{b}_k,$$

$$a_i,\ \overline{a}_i \in A,\ b_i,\ \overline{b}_i \in B.$$

By using (3) and (3′) conclude that $x$, $y$ are congruent with the product of elements

$$a,\ a \in A,\ b,\ b' \in B,\ z \in G,\ \langle b,\ a\rangle^{z},\ \langle a,\ a'\rangle^{z},\ \langle b,\ b'\rangle^{z},\ \langle a,\ b\rangle^{z},$$

modulo $B(G)$.

Every element of this form with the following relations can be written as the product of element without $z$ power.

(5′) $\langle a,\ a'\rangle^{a_0} = \langle a^{a_0},\ a'^{a_0}\rangle$.

(6′) $\langle a,\ a'\rangle^{b_0} \sim \langle b_0,\ [a,\ a']\rangle\langle a,\ a'\rangle$.

(12) $\langle a,\ b\rangle^{a_0} \sim \langle a_0 a,\ b\rangle\langle b,\ a_0\rangle$.

(13) $\langle a,\ b\rangle^{b_0} \sim \langle b_0,\ a\rangle\langle a,\ b_0 b\rangle$.

By changing $a$ to $b$, $a_0$ to $b_0$, and $a'$ to $b'$, we get the similar formula as above. Relation (5′) comes from (5), (4) and (12) from (3), also (3′) is obtained by (2). Therefore, every $w \in \langle G,\ G\rangle$ is the congruent with a product $\pi$ of terms $\langle a,\ a'\rangle$, $\langle b,\ b'\rangle$, $\langle a,\ b\rangle$, $\langle b,\ a\rangle$.

Now, by (8), (9), (10) commute each term like $\langle b,\ b'\rangle$. Begin this process with the term in the right hand side and commute term by term. Therefore, for every $w \in \langle G,\ G\rangle$

$$w \sim \pi \sim \pi'\beta,$$

where $\beta$ is the product of $\langle b,\ b'\rangle$, $s$ and $\pi'$ the product of $\langle a,\ a'\rangle$, $\langle a,\ b\rangle$. Now every term like $\langle a,\ a'\rangle$ in $\pi'$ by dual of (8), (9) (i.e., using inverse relations and replacing $a$, $b$) remove it to the left,

$$w \sim \pi'\beta \sim \alpha\pi''\beta,$$

where $\pi''$ only consists of $\langle a,\ b\rangle$, $\langle a,\ b\rangle$ and $\alpha$ the product of $\langle a,\ a'\rangle$. Replacing $\langle b,\ a\rangle$ by $\langle a,\ b\rangle^{-1}$ in $\pi'$ we exchange $\pi''$ by $\mu \in \mathcal{M}$. Therefore, $\mu \in \mathcal{M}$, $\alpha \in \mathcal{B}$, $\alpha \in \mathcal{A}$, $w \sim \alpha\mu\beta$.

Let $w \in Z(G)$, i.e., $[w] = 1$. So $[\alpha][\mu][\beta] = [w] = 1$. Now $[\alpha] = 1$, since there is a map into $A$, similarly $[\beta] = 1$. Therefore, $[\mu] = 1$. On the other hand, $[\mu] = 1$ implies $\mu = 1 \in \mathcal{M} \subset \langle G,\ G\rangle$. For if $\mu$ is a reduced word in

the free group $\mathcal{M}$, then

$$\mu = \langle a_1, b_1\rangle^{\varepsilon_1}\langle a_2, b_2\rangle^{\varepsilon_2}\cdots\langle a_p, b_p\rangle^{\varepsilon_p},\ \varepsilon_i = \pm 1,\ a_i \neq 1,\ b_i \neq 1.$$

By induction on $p$ we write $[\mu]$ as a reduced word in $G = A * B$. The last two entries of this word are $b_p^{-1}a_p^{-1}$ if $\varepsilon_p \pm 1$ and $a_p^{-1}b_p^{-1}b_p^{-1}$ if $\varepsilon_p = 1$. In fact, if $\mu$ is nonempty and $[\mu] \neq 1$, then for

$$\mu = 1,\ w \sim \alpha\mu\beta = \alpha\beta,$$

where $[\alpha] = 1$, $[\beta] = 1$.

So $H(G) = i_*H(A)\, j_*H(B)$.

Every topological group is the quotient of a free topological group [3]. Let $R \xrightarrow{i} F \xrightarrow{\pi} G$ be an extension of $G$ by $F$ with a continuous section, i.e., a continuous map $u : G \to F$ such that $\pi u$ is the identity.

Define $\langle G, G\rangle \to F$ by $\langle x, y\rangle \mapsto [u(x), u(y)]$.

The definition is independent of the choice of $x$, $y$ because $R$ is in the center of $F$. Under this map the image of $Z(G)$ is $R \cap [F, F]$ and for $B(G)$ is 1. Hence we have the following continuous map

$$\phi : H(G) \to R \cap [F, F].$$

It can be shown that the following sequence is exact at $H(G)$, i.e., $\ker\phi = \operatorname{Im}\eta_*$

$$H(F) \xrightarrow{\eta_*} H(G) \xrightarrow{\phi} R \cap [F, F].$$

Let $R^0 = R/\overline{[F, F]}$, $F^0 = F/\overline{[R, F]}$. Then

$$\begin{array}{ccccc} F & \xrightarrow{\lambda} & F^0 & \xrightarrow{\eta} & G \\ \cup & & \cup & & \\ R & \to & R^0 & & \\ \cup & & & & \\ [F, R], & & & & \end{array}$$

where $\lambda$ and $\eta$ are the quotient maps. The subgroup $R^0$ is in the center of $F^0$. Therefore, $\phi : H(G) \to R^0 \cap [F^0, F^0]$, is onto. Since the sequence

$$H(F^0) \to H(G) \to R^0 \cap [F^0, F^0]$$

is exact, if $\eta_* = 0$, then $\phi$ is one-one.

For if

$$w = \langle x_1, y_1\rangle \langle x_2, y_2\rangle \dots \langle x_p, y_p\rangle \in Z(F^0),$$

then

$$[w] = [x_1, y_1] \dots [x_p, y_p] = 1 \in F^0.$$

If we choose $\overline{x}_i = u(x_i)$, $\overline{y}_i = u(y_i) \in F$ such that $\lambda(\overline{x}_i) = x_i$, $\lambda(\overline{y}_i) = y_i$, then

$$\overline{w} = \langle \overline{x}_1, \overline{y}_1\rangle \langle \overline{x}_2, \overline{y}_2\rangle \dots \langle \overline{x}_p, \overline{y}_p\rangle,$$

$$\lambda_\# \overline{w} = w, \quad [\overline{w}] = [w] = 1,$$

so

$$[\overline{w}] \in \overline{[F, R]}.$$

If $[\overline{w}] \in [F, R]$, then

$$[\overline{w}] = [f_1, r_1][f_2, r_2] \dots [f_q, r_q], \; f_i \in F, \; r_i \in R.$$

Since $F$ is a free topological group and $H(F) = 0$, so $Z(F) = B(F)$. Therefore

$$\widetilde{w} \sim \langle f_1, r_1\rangle \langle f_2, r_2\rangle \dots \langle f_q, r_q\rangle$$

$$\begin{aligned} \eta_\# \overline{w} &= \eta_\# \lambda_\# \, [\overline{w}] \\ &\sim \eta_\# \lambda_\# \, \langle f_1, r_1\rangle \dots \langle f_q, r_q\rangle \\ &\sim \langle \eta\lambda f_1, 1\rangle \dots \langle \eta\lambda f_q, 1\rangle \\ &\sim 1. \end{aligned}$$

Hence $\eta_* = 0$. If $[\overline{w}]$ is a limit point of $[R, F]$, then $\lambda_\#[\overline{w}] = 1$, since $\lambda_\#$ is a continuous homomorphism. So $\eta_\#\lambda_\#[\overline{w}] = 1$, i.e., $\eta_* = 0$. Therefore $\phi : H(G) \to R^0 \cap [F^0, F^0]$, is a one-one continuous homomorphism. Note that

$$R^0 \cap [F^0, F^0] = R \cap [F, F]/\overline{[R, F]}$$

which is the Schur multiplicator of topological group.

**The Main Theorem**: *There is a one-one, onto continuous homomorphism from $H(G)$ to $R \cap [F, F]/\overline{[R, F]}$, the Schur multiplicator of a separable topological group $G$.*

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# TYPES AND ENUMERATION OF IDEMPOTENT MATRICES

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## Abstract

In this paper, we determine all the idempotent Boolean matrices and their types which are the sums of 4 distinct cells. We also give the algorithm to enumerate $3 \times 3$ and $4 \times 4$ idempotent Boolean matrices and obtain 123 and 2360 idempotent matrices, respectively.

## 1. Introduction

There are many papers on the study of idempotent matrices. Boolean matrices also have been the subject of research by many authors because of their association with nonnegative real matrices ([1]-[3]). But there are few papers on the enumeration of idempotent Boolean matrices. In [2], Beasley and Pullman studied on the idempotent matrices and their preservers.

In this paper, we consider the following questions: How many idempotent matrices are there among the $n \times n$ Boolean matrices? And what are their forms?

In Section 2, we give some definitions and some preliminaries. In Section 3, we determine the forms of all the idempotent matrices with

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four or less nonzero entries. In Section 4, we give the algorithm which enables us to determine all $3 \times 3$ and $4 \times 4$ idempotent matrices and obtain 123 and 2360 idempotent matrices, respectively.

## 2. Definitions and Preliminaries

Let $\mathbb{B} = \{0, 1\}$ be the (binary) Boolean algebra equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1 + 1 = 1$. Let $\mathcal{M}_n(\mathbb{B})$ denote the set of $n \times n$ matrices with entries in $\mathbb{B}$.

In this paper, all matrices are $n \times n$ Boolean matrices with entries in $\mathbb{B}$. The zero matrix is denoted by $O$, the identity matrix by $I$ and the matrix with all entries equal to 1 is denoted by $J$.

A zero-one $n \times n$ matrix with only one entry equal to 1 is called a *cell*. If the nonzero entry occurs in row $i$ and column $j$, we denote the cell by $E_{ij}$ and say that the cell is in row $i$ and it is in column $j$. A line is a row or column. A set of cells is *collinear* if they are all in the same line. When $i \neq j$, we say $E_{ij}$ is an *off-diagonal cell*; $E_{ii}$ is a *diagonal cell*.

The following two lemmas are immediate consequences of the rules of matrix multiplication and were used in [2].

**Lemma 2.1.** *For all indices $i$, $j$, $u$, and $v$, $E_{ij}E_{uv} = E_{iv}$ or $O$ according as $j = u$ or $j \neq u$.*

**Lemma 2.2.** *Suppose $C$ and $D$ are cells and $CD \neq O$.*

(a) *If $C$ and $D$ are diagonal, then $C = D$.*

(b) *If $C$ is a diagonal cell and $D$ is not, then $CD = D$, and $C$ and $D$ are in the same row.*

*If $D$ is a diagonal cell and $C$ is not, then $CD = C$, and $C$ and $D$ are in the same column.*

(c) *If $C$ and $D$ are off-diagonal cells, then either*

(i) *$CD$ is an off-diagonal cell distinct from $C$ and $D$, and $DC = O$*

*or*

(ii) $D = C^T$, *and* $CD$ *and* $DC$ *are distinct diagonal cells.*

A matrix $E$ is called *idempotent* if $E^2 = E$. Otherwise, $E$ is called *nonidempotent*.

Notice that all diagonal cells are idempotent but all off-diagonal cells are nonidempotent in $\mathcal{M}_n(\mathbb{B})$. Furthermore, any sums of diagonal cells are idempotent.

**Proposition 2.3.** *Suppose* $E$ *is a diagonal cell and* $F$ *is an off-diagonal cell. Then their sum is idempotent if and only if they are collinear.*

**Proof.** ($\Rightarrow$) Suppose $(E + F)^2 = E + F$. By Lemma 2.1, $E^2 = E$ and $F^2 = O$. Therefore $E + EF + FE = E + F$ and so $EF + FE = F$. By Lemma 2.2(b), $E$ and $F$ are collinear.

($\Leftarrow$) Without loss of generality, we assume that $E = E_{ii}$ and $F = E_{ij}$ with $i \neq j$. Then

$$\begin{aligned}(E + F)^2 &= (E_{ii} + E_{ij})^2 \\ &= E_{ii}^2 + E_{ii}E_{ij} + E_{ji}E_{ii} + E_{ij}^2 \\ &= E_{ii} + E_{ij} + O + O = E + F.\end{aligned}$$

Thus $E + F$ is idempotent.

**Proposition 2.4.** *Suppose* $E$ *and* $F$ *are distinct off-diagonal cells. Then their sum is not idempotent.*

**Proof.** If $E$ and $F$ are in the same row, say $E = E_{ij}$ and $F = E_{ik}$ with $i \neq j, k$ and $j \neq k$, then $(E + F)^2 = O$ and so $E + F$ is not idempotent. If $E$ and $F$ are in the same column, say $E = E_{ij}$ and $F = E_{kj}$ with $i \neq j, k$ and $j \neq k$, then $(E + F)^2 = O$ and so $E + F$ is not idempotent. If $E$ and $F$ are not in the same line, say $E = E_{ij}$ and

$F = E_{kl}$ with $i \neq j, k$ and $k \neq l$, then $(E + F)^2 = O$ or $E_{il}$ or $E_{kj}$ or $E_{il} + E_{kj}$ according as $(j \neq k$ and $l \neq i)$ or $(j = k$ and $l \neq i)$ or $(j \neq k$ and $l = i)$ or $(j = k$ and $l = i)$ and so $E + F$ is not idempotent.

From the same method as in the proof of Proposition 2.4, we obtain the general result.

**Corollary 2.4.1.** *Suppose $E_1, E_2, \ldots, E_k$ are mutually distinct off-diagonal cells. Then their sum is not idempotent.*

**Proposition 2.5.** *Suppose E, F, and G are mutually distinct cells, E and F are diagonal but G is not. Then their sum is idempotent if and only if G is in the same line to E or F.*

**Proof.** The necessity is immediate and so we only prove the sufficiency. Suppose $(E + F + G)^2 = E + F + G$. Then by Lemma 2.1, $E^2 = E$, $F^2 = F$, $EF = FE = O$ and $G^2 = O$. So we have

$$E + F + (EG + GE) + (FG + GF) = E + F + G. \tag{2.1}$$

Notice that $EG + GE = O$ or $G$, and $FG + GF = O$ or $G$. Therefore, we obtain that the equation (2.1) implies that $(EG + GE) + (FG + GF) = G$. Thus we have $EG + GE = G$ or $FG + GF = G$. By Lemma 2.2(b), $E$ and $G$ are in the same line or $F$ and $G$ are in the same line.

We can extend the Proposition 2.5 to the case of many diagonal cells.

**Corollary 2.5.1.** *Suppose $E_1, E_2, .., E_k$, and F are mutually distinct cells, $E_i$'s are diagonal but F is not. Then their sum is idempotent if and only if F is in the same line to at least one of $E_1, E_2, .., E_k$.*

**Proposition 2.6** [2]. *Suppose E, F, and G are mutually distinct cells, E is diagonal but F and G are not. Then their sum is idempotent if and only if they are collinear.*

Thus we have obtained the forms of idempotent matrices with at most 3 nonzero entries. Next, we consider the forms of idempotent matrices with 4 nonzero entries.

## 3. Types of Idempotent Matrices with 4 Nonzero Entries in $\mathcal{M}_n(\mathbb{B})$

In this section, we classify completely the matrices of the sums of mutually distinct four cells and obtain the cases of being idempotent.

**Lemma 3.1.** *Suppose* $E_1$, $E_2$, $E_3$, *and* $E_4$ *are mutually distinct diagonal cells in* $\mathcal{M}_n(\mathbb{B})$ *with* $n \geq 4$. *Then their sum is idempotent.*

**Proof.** It is trivial.

**Lemma 3.2.** *Suppose* $E_1$, $E_2$, $E_3$, *and* $F_1$ *are mutually distinct cells,* $E_1$, $E_2$, *and* $E_3$ *are diagonal but* $F_1$ *is not. Then their sum is idempotent if and only if* $F_1$ *is in the same line to at least one of* $E_1$, $E_2$ *or* $E_3$.

**Proof.** This is a special case of Corollary 2.5.1.

**Theorem 3.3.** *Suppose* $E$, $F$, $G$, *and* $H$ *are mutually distinct cells,* $E$ *and* $F$ *are diagonal but* $G$ *and* $H$ *are not. Then their sum is idempotent if and only if they satisfy one of the following conditions*:

(1) *$G$ is in the same line to each $E$ and $F$ and* $H = G^T$.

(2) *$G$ and $H$ are collinear and they are in the same line to $E$ or $F$.*

(3) *$G$ and $H$ are not collinear with* $GH = HG = O$ *and $G$ is in the same line to either $E$ or $F$ and $H$ is so.*

**Proof.** The necessity is immediate and so we only prove the sufficiency. Suppose $(E + F + G + H)^2 = E + F + G + H$. By Lemma 2.1, $E^2 = E$, $F^2 = F$, $G^2 = H^2 = O$, and $EF = FE = O$. Thus we have

$$E + F + (EG + GE) + (EH + HE) + (FG + GF)$$
$$+ (FH + HF) + (GH + HG) = E + F + G + H. \qquad (3.1)$$

Notice that $EG + GE = O$ or $G$, $EH + HE = O$ or $H$, $FG + GF = O$ or $G$, and $FH + HF = O$ or $H$. First we suppose that $GH + HG \neq O$. Then $GH \neq O$ or $HG \neq O$, say $GH \neq O$. By Lemma 2.2(a), $H = G^T$ and $GH$ and $HG$ are distinct diagonal cells. Therefore, we obtain that the

equation (3.1) implies that $GH + HG = E + F$. Without loss of generality, we assume that $GH = E$ and $HG = F$. Let $E = E_{ii}$ and $F = E_{jj}$ with $i \neq j$. Then $G$ is of the form $E_{ik}$ with $i \neq k$ because $GH = E$. Similarly, $H = E_{jt}$ with $j \neq t$. Since $H = G^T$, $E_{jt} = E_{ki}$ and so $j = k$ and $t = i$. That is, we obtain that $E = E_{ii}$, $F = E_{jj}$, $G = E_{ij}$, and $H = E_{ji}$ which satisfy the condition (1). Next, we suppose that $GH + HG = O$. Then we have

$$(EG + GE) + (EH + HE) + (FG + GF) + (FH + HF) = G + H.$$

Notice that ($EG + GE = G$ or $FG + GF = G$) and ($EH + HE = H$ or $FH + HF = H$). Without loss of generality, we assume that $E = E_{ii}$ and $F = E_{jj}$ with $i \neq j$.

We prove this theorem in three steps.

**Case 1.** Assume that $EG + GE = G$ and $FG + GF = G$.

By Lemma 2.2(b), $E$ and $G$ are in the same line, and $F$ and $G$ are in the same line. Thus $G$ is in the same line to each $E$ and $F$. Therefore, the form of $G$ is either $E_{ij}$ or $E_{ji}$.

**Case 1.1.** $EH + HE = H$ and $FH + HF = H$.

By Lemma 2.2(b), $E$ and $H$ are in the same line, and $F$ and $H$ are in the same line. So $H$ is in the same line to each $E$ and $F$. Thus the form of $H$ is either $E_{ji}$ or $E_{ij}$ according to $G = E_{ij}$ or $G = E_{ji}$. Therefore, $GH + HG = E + F(\neq O)$ which is a contradiction.

**Case 1.2.** $EH + HE = H$ and $FH + HF = O$.

By Lemma 2.2(b), $E$ and $H$ are in the same line, and $F$ and $H$ are not in the same line. Thus $H$ is of the form $E_{ik}$ or $E_{ti}$ with $i \neq k, t$. If $G = E_{ij}$, then $H$ is of the form $E_{ik}$ with $k \neq j$. (If not, $H = E_{ti}$ and $t \neq j$ and so $HG = E_{tj}(\neq O)$ which is a contradiction.) Therefore $E$, $G$, and $H$ are in the same column. If $G = E_{ji}$, then $H$ is of the form $E_{ti}$ with $t \neq j$. (If not, $H = E_{ik}$ and $k \neq j$ and so $GH = E_{tk}(\neq O)$ which is a contradiction.) Therefore $E$, $G$, and $H$ are in the same row.

**Case 1.3.** $EH + HE = O$ and $FH + HF = H$.

By the similar method of Case 1.2, $F$, $G$, and $H$ are in the same line.

**Case 2.** Assume that $EG + GE = G$ and $FG + GF = O$.

By Lemma 2.2(b), $E$ and $G$ are in the same line, and $F$ and $G$ are not in the same line. So the form of $G$ is either $E_{ik}$ or $E_{ti}$ with $i \neq k, t$ and $j \neq k, t$.

**Case 2.1.** $EH + HE = H$ and $FH + HF = H$.

By the similar method of Case 1.2, $E$, $G$, and $H$ are in the same line.

**Case 2.2.** $EH + HE = H$ and $FH + HF = O$.

By Lemma 2.2(b), $E$ and $H$ are in the same line, and $F$ and $H$ are not in the same line. So $H$ is of the form either $E_{ia}$ or $E_{bi}$ with $i \neq a, b$ and $j \neq a, b$.

(a) $G = E_{ik}$ and $H = E_{ia}$.

We notice that $k \neq a$ because $G$ and $H$ are distinct. Thus $E$, $G$, and $H$ are in the same row.

(b) $G = E_{ik}$ and $H = E_{bi}$.

Now $HG = E_{bi}E_{ik} = E_{bk}$. But this cell is distinct from $G$ and $H$. This contradicts to $GH + HG = O$.

(c) $G = E_{ti}$ and $H = E_{ia}$.

Now $GH = E_{ti}E_{ia} = E_{ta}$. But this cell is distinct from $G$ and $H$. This contradicts to $GH + HG = O$.

(d) $G = E_{ti}$ and $H = E_{bi}$.

We notice that $k \neq b$ because $G$ and $H$ are distinct. Thus $E$, $G$, and $H$ are in the same column.

From (a), (b), (c) and (d), $E$, $G$, and $H$ are in the same line.

**Case 2.3.** $EH + HE = O$ and $FH + HF = H$.

. By Lemma 2.2(b), $E$ and $H$ are not in the same line, and $F$ and $H$ are in the same line. Thus $H$ is of the form either $E_{ja}$ or $E_{bj}$ with $a \neq i, j$ and $b \neq i, j$.

(e) $G = E_{ik}$ and $H = E_{ja}$.

Since $GH = HG = O$, $k \neq j$ and $a \neq i$. Thus $G$ is in the same row only to $E$ and $H$ is in the same row only to $F$. If $a = k$, then they satisfy the condition (2). If $a \neq k$, then they satisfy the condition (3).

(f) $G = E_{ik}$ and $H = E_{bj}$.

Notice that $k \neq b$. (If $k = b$, then $GH = E_{ij} (\neq O)$ which is a contradiction.) Thus $G$ is in the same row only to $E$ and $H$ is in the same column only to $F$. Therefore, they satisfy the condition (3).

(g) $G = E_{ti}$ and $H = E_{ja}$.

Notice that $t \neq a$. (If $t = a$, then $HG = E_{ji} (\neq O)$ which is a contradiction.) Thus $G$ is in the same column only to $E$ and $H$ is in the same row only to $F$. Therefore, they satisfy the condition (3).

(h) $G = E_{ti}$ and $H = E_{bj}$.

Since $GH + HG = O$, $b \neq i$ and $t \neq j$. Thus $G$ is in the same column only to $E$ and $H$ is in the same column only to $F$. If $b = t$, then they satisfy the condition (2). If $b \neq t$, then they satisfy the condition (3).

**Case 3.** Assume that $EG + GE = O$ and $FG + GF = G$.

The proof is similar to Case 2.

Let $C$ be a matrix in $\mathcal{M}_n(\mathbb{B})$. Then we call $C$ to be a *rectangle form* if $C$ has only four 1's and the four 1's constitute a rectangle with a 1 on diagonal and the other three 1's on off-diagonal.

**Theorem 3.4.** *Suppose $E$, $G$, $H$, and $K$ are mutually distinct cells, $E$ is diagonal but $G$, $H$, and $K$ are not. Then their sum is idempotent if and only if they satisfy one of the following conditions*:

(1) *They are collinear.*

(2) *They have the rectangle form.*

**Proof.** The necessity is immediate and so we only prove the sufficiency. Suppose $(E + G + H + K)^2 = E + G + H + K$. By Lemma 2.1, $E^2 = E$ and $G^2 = H^2 = K^2 = O$. Thus we have

$$\begin{aligned} &E + (EG + GE) + (EH + HE) + (EK + KE) \\ &\quad + (GH + HG) + (GK + KG) + (HK + KH) \\ &= E + G + H + K. \end{aligned} \tag{3.2}$$

We notice that $EG + GE = O$ or $G$, $EH + HE = O$ or $H$, and $EK + KE = O$ or $K$. First we show that $GH + HG = O$ or $K$. Suppose $GH + HG \neq O$. Then $GH \neq O$ or $HG \neq O$, say $GH \neq O$. By Lemma 2.2(a), $GH$ is an off-diagonal cell distinct from $G$ and $H$ with $HG = O$. Thus the equation (3.2) implies that $FG = H$ which is the desired result. Similarly, $GK + KG = O$ or $H$ and $HK + KH = O$ or $G$.

**Case 1.** Assume that $GH + HG = O$.

Without loss of generality, we assume that $E = E_{ii}$.

**Case 1.1.** $GK + KG = O$ and $HK + KH = O$.

We notice that the equation (3.2) implies that $EG + GE = G$, $EH + HE = H$, and $EK + KE = K$. By Lemma 2.2(b), $E$ and $G$ are collinear, $E$ and $H$ are collinear, and $E$ and $K$ are collinear. Thus $G$ is of the form $E_{ia}$ or $E_{bi}$. Similarly, $H = E_{ic}$ or $E_{di}$ and $K = E_{ie}$ or $E_{fi}$ with $i \neq a, b, c, d, e,$ and $f$.

(a) $G = E_{ia}$, $H = E_{ic}$, and $K = E_{ie}$.

Since $G$, $H$, and $K$ are mutually distinct cells, $i$, $a$, $c$, and $e$ are mutually distinct. Therefore $E$, $G$, $H$, and $K$ are in the same row.

(b) $G = E_{ia}$, $H = E_{ic}$, and $K = E_{fi}$.

Now $KG = E_{fa} (\neq O)$ which is impossible.

(c) $G = E_{ia}$, $H = E_{di}$, and $K = E_{ie}$.

Now $HK = E_{de}(\neq O)$ which is impossible.

(d) $G = E_{ia}$, $H = E_{di}$, and $K = E_{fi}$.

Now $KG = E_{fa}(\neq O)$ which is impossible.

If $G = E_{bi}$, then by the above method, $E$, $G$, $H$, and $K$ are in the same column.

**Case 1.2.** $GK + KG = H$ and $HK + KH = O$.

We notice that the equation (3.2) implies that $EG + GE = G$, $EH + HE = O$ or $H$, and $EK + KE = K$. By Lemma 2.2(b), $E$ and $G$ are collinear, and $E$ and $K$ are collinear. Thus $G$ is of the form $E_{ia}$ or $E_{bi}$, and $K$ is of the form $E_{ic}$ or $E_{di}$ with $i \neq a, b, c,$ and $d$. Since $GK + KG = H$, ($GK = H$ and $KG = O$) or ($GK = O$ and $KG = H$).

(e) $GK = H$ and $KG = O$.

Since $KG = O$, $G$ and $K$ are of the forms ($G = E_{ia}$ and $K = E_{ic}$) or ($G = E_{bi}$ and $K = E_{ic}$) or ($G = E_{bi}$ and $K = E_{di}$). Let $G = E_{ia}$ and $K = E_{ic}$. Since $GK = H$, $a = i$ which is impossible. Let $G = E_{bi}$ and $K = E_{ic}$. Since $GK = H$ and $H$ is an off-diagonal cell, $H = E_{bc}$ and $b \neq c$. Since $i \neq b, c$, $EH + HE = O$. Thus $E$, $G$, $H$, and $K$ have the rectangle form. Let $G = E_{bi}$ and $K = E_{di}$. Since $GK = H$, $d = i$ which is impossible.

(f) $GK = O$ and $KG = H$.

The proof is similar to the above (e).

**Case 1.3.** $GK + KG = O$ and $HK + KH = G$.

By the similar method of Case 1.2, $E$, $G$, $H$, and $K$ have the rectangle form.

**Case 1.4.** $GK + KG = H$ and $HK + KH = G$.

We notice that the equation (3.2) implies that $EK + KE = K$. By Lemma 2.2(b), $E$ and $K$ are collinear. Thus $K$ is of the form $E_{ia}$ or $E_{bi}$ with $i \neq a, b$. Now, we will only consider $K = E_{ia}$, $GK = H$, and $HK = G$. Since $GK = H$ and $HK = G$, $KG = KH = O$. Since $K = E_{ia}$ and $GK = H$, $G$ is of the form $E_{ci}$ and $H$ is of the form $E_{ca}$ with $i \neq c$ and $c \neq a$. Since $HK = G(\neq O)$, $a = c$ which is impossible.

**Case 2.** Assume that $GH + HG = K$.

The proof is similar to Case 1.

## 4. Algorithm for the Enumeration of $3 \times 3$ and $4 \times 4$ Idempotent Matrices

In this section, we give the algorithm for the enumeration of $3 \times 3$ and $4 \times 4$ idempotent matrices. By the following algorithm, we have 2360 idempotent matrices among 4-square $2^{16} = 65536$ Boolean matrices.

### Count Idempotent Matrices (FoxPlus for DOS)

```
set talk off
set decimal to 0
dime A(5, 5), B(5, 5)
Matrix = 4
Components = Matrix **2
Y = 0
set alte to TmpMat
set alte on
N = 0
do while N <= 2 ** Components - 1
X = N
T =''
```

```
Tr = replicate ('0', Components)
do while X > 0
R = X - 2 * int(X/2) + 1
T = substr ("01", R, 1) + T
X = int(X/2)
enddo
Tr = left(Tr, Components-len(T)) + T
i = 1
do while i <= Matrix
j = 1
do while j <= Matrix
A(i, j) = val(substr(Tr, (i - 1) * Matrix + j, 1))
B(i, j) = 0
j = j + 1
enddo
i = i + 1
enddo
IfExit = .T.
i = 1
do while i <= Matrix
if .not. IfExit
exit
endif
j = 1
do while j <= Matrix
```

```
k = 1
do while k <= Matrix
B(i, j) = A(i, k) * A(k, j) + B(i, j)
B(i, j) = iif(B(i, j) > 1, 1, B(i, j))
k = k + 1
enddo
if B(i, j)# A(i, j)
IfExit = .F.
exit
endif
j = j + 1
enddo
i = i + 1
enddo
if IfExit
Y = Y + 1
?
i = 1
do while i <= Matrix
j = 1
?
do while j <= Matrix
?? transform (A(i, j),' 99')
j = j + 1
enddo
```

```
i = i + 1
enddo
endif
N = N + 1
enddo
set alte off
? Y
set talk on
return
```

For all the 4-square idempotent matrices, they can be written on so many pages. The complete details of this idempotent matrices can be obtained from the first author. By the same algorithm with "Matrix = 3", we have 123 idempotent matrices among 3-square $2^9 = 512$ Boolean matrices.

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# BOOK REVIEW

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